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Rescaled weighted random ball models and stable self-similar random fields

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Abstract

We consider weighted random balls in $\mathbb{R}^d$ distributed according to a random Poisson measure with heavy-tailed intensity and study the asymptotic behavior of the total weight of some configurations in $\mathbb{R}^d$ while we perform a zooming operation. The resulting procedure is very rich and several regimes appear in the limit, depending on the intensity of the balls, the zooming factor, the tail parameters of the radii and the weights. Statistical properties of the limit fields are also evidenced, such as isotropy, self-similarity or dependence. One regime is of particular interest and yields α-stable stationary isotropic self-similar generalized random fields which recovers Takenaka fields, Telecom process or fractional Brownian motion.

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0. Introduction

In this work, we consider the so-called weighted random ball model and investigate its convergence when suitably rescaled and normalized. We exhibit three different asymptotic regimes driving the macroscopic and microscopic variations of this model, namely (i) a stable, translation and rotation invariant, self-similar random field on $\mathbb{R}^d$, (ii) a Poissonian field and (iii) a stable field with independence. The weighted random ball model is constructed in the

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following way: the centers of the balls are distributed according to a Poisson point process, with each center x labelled with a random radius r and a random weight m. The field under study is, roughly speaking, at each point, the weight density defined as the sum of the weights of the balls containing this point. The overlap of the balls yields non-trivial spatial correlations when the random radii of the balls are heavy tailed.

This fairly simple geometric construction has found numerous applications and is pertinent in various modeling situations. Similar stochastic models were considered by Kaj in [6] when modeling a simplified wireless network that consists of a collection of spatially distributed stations equipped with emitters for transmission over a common communication channel. Here, the location of a station or of a network node is represented by the point x, its range by the radius r and its power by the weight m. The weight density measures the total power of emission at a given point and in this case, m is supposed to be non-negative. But our model supports more generally real-valued weights.

In [1], Biermé and Estrade consider similar models in dimension $d = 2$ as models in imagery (in this case, the weight intensity stands for the gray level of a pixel in a black and white picture) and in dimension $d = 3$ for modeling three-dimensional porous or heterogeneous media (here, the weight density is seen as a mass density). They investigate the microscopic properties of the random ball configurations by performing a scaling operation which amounts to zoom in smaller regions of space. In [7], Kaj et al. study similar random grain model by shrinking to zero the volume of the grains. This amounts to analyse the macroscopic properties of the random ball configurations by performing a scaling operation which amounts here to zoom out over larger areas.

Recently, Biermé, Estrade and Kaj introduce in [2] a general framework for rescaled random ball model allowing both zoom-in (as in [1]) and zoom-out (as in [7]). In this zooming procedure, several limit fields arise, which are either of Gaussian or of Poisson type according to the respective asymptotic of the zooming rate and of the Poisson intensity of the balls. Furthermore, they show that essentially all Gaussian, translation and rotation invariant self-similar generalized random fields can be obtained as such a limit.

Note that in the rescaled random ball model of [1,7] and [2], the weights in the field under study are fixed equal to $m \equiv 1$. Models with randomized weights have been less intensively studied. In dimension $d = 1$, Kaj and Taqqu study in [4] limiting schemes for weighted random ball model, deriving Gaussian, Poisson and stable regimes. This model applies in particular to study the random variation in packet networks computer traffic.

Our main contribution in this paper is to introduce a general study of macroscopic and microscopic variations in weighted models in $\mathbb{R}^d$. This generalizes both [2] since the balls are randomly weighted and [4] since we consider an arbitrary dimension d and more general configurations on the balls. As in [7] and [4], three different regimes appear according to the relative behavior of the scaling rate and of the Poisson intensity. In particular, when the random weights are heavy tailed, the limit generalized random fields are stable, translation and rotation invariant, and also self-similar. The paper is organized as follows. The model under study is described in Section 1. Our main results under different scaling regimes are stated and discussed in Section 2. Finally, Section 3 is devoted to the proof of technical lemmas and of the main results.

1. Model of weighted random balls

We consider random balls $B(x, r) = \{y \in \mathbb{R}^d : \|y - x\| < r\}$ with weight m, the triplet (x, r, m) being distributed according to a Poisson random measure $N_\lambda(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m)$ on $\mathbb{R}^d \times$

$\mathbb{R}^+ \times \mathbb{R}$ with intensity

$$n(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m) = \lambda \mathrm{d}x\, F(\mathrm{d}r) G(\mathrm{d}m)$$

where λ is positive, F is a positive measure on $\mathbb{R}^+$ and G a probability measure on $\mathbb{R}$. Here and in what follows, $\|\cdot\|$ stands for the usual Euclidean norm on $\mathbb{R}^d$.

The point process of the centers of the balls in $\mathbb{R}^d$ is the projection of the point process in $\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}$ corresponding to the Poisson random measure $N_\lambda(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m)$. It is easily seen that it is a Poisson point process with intensity $\lambda \mathrm{d}x$, and hence the parameter λ is interpreted as the intensity of the balls in $\mathbb{R}^d$.

We suppose that the measure F driving the distribution of the radius r is absolutely continuous $F(\mathrm{d}r) = f(r)\mathrm{d}r$ with

$$\int_{\mathbb{R}^+} r^{\mathrm{d}} F(\mathrm{d}r) < +\infty \tag{1}$$

and such that for either $\epsilon = +1$ or $\epsilon = -1$,

$$f(r) \sim_{r \to 0^\epsilon} C_\beta r^{-1-\beta} \tag{2}$$

where by convention $0^{+1} = 0$ and $0^{-1} = +\infty$. As will be explained later, the case $\epsilon = +1$ will be referred as the zoom-in case, whereas the case $\epsilon = -1$ will be referred as the zoom-out case. Condition (2) assumes a power behavior of the radius density at the origin (zoom-in case $\epsilon = +1$) or at infinity (zoom-out case $\epsilon = -1$). Condition (1) is equivalent to the finiteness of the volume of the random balls. Note that assumptions (1) and (2) together imply that for $\epsilon = +1$, we must have $\beta < d$, while for $\epsilon = -1$, we must have $\beta > d$.

We suppose that the probability law G belongs to the normal domain of attraction of the α-stable distribution $S_\alpha(\sigma, b, \tau)$ with $\alpha \in (1, 2]$, i.e. if $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) according to G, $n^{-1/\alpha}(X_1 + \cdots + X_n) \Rightarrow S_\alpha(\sigma, b, \tau)$. We recall the following estimate (see [3]) of the characteristic function φ_G of G as $\theta \to 0$

$$\varphi_G(\theta) = 1 + \mathrm{i}\theta\tau - \sigma^\alpha |\theta|^\alpha (1 + \mathrm{i}b\varepsilon(\theta)) \tan(\pi\alpha/2) + o(|\theta|^\alpha), \tag{3}$$

where here, and in what follows, $\varepsilon(a) = +1$ if $a > 0$, $\varepsilon(a) = -1$ if $a < 0$ and $\varepsilon(0) = 0$. In case $\alpha \in (1, 2)$, typical choices for G are heavy-tailed distributions while for $\alpha = 2$, G is any distribution with finite variance. In this latter case, we recover a weighted version of the main results in [2] (set $G = \delta_1$ to recover exactly the setting described in [2]).

Let $\mathcal{M}$ denote the set of signed measures on $\mathbb{R}^d$ with finite total variation $|\mu|(\mathbb{R}^d)$, where $|\mu|$ is the total variation of a measure μ. We recall that equipped with the norm of total variation $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}^d)$, $\mathcal{M}$ is a Banach space. We consider the random field

$$M(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m\mu(B(x, r)) N_\lambda(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m) \tag{4}$$

indexed by signed measures $\mu \in \mathcal{M}$. When $\mu = \delta_y$, $y \in \mathbb{R}^d$, $M(\delta_y)$ is the weight density at point y as described in the introduction: it is the sum of the algebraic weights of the balls containing the point y.

Note that the stochastic integral in (4) is well defined and has finite expected value since

$$\int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} |m\mu(B(x,r))|n(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m) \le \int_{\mathbb{R}} |m|G(\mathrm{d}m) \times \lambda|B(0,1)||\mu|(\mathbb{R}^d)\int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < +\infty$$

where $|A|$ stands for the Lebesgue measure of a Borel set A. Furthermore, the expected value is given by

$$\mathbb{E}[M(\mu)] = \lambda|B(0,1)|\int_{\mathbb{R}} mG(\mathrm{d}m)\int_{\mathbb{R}^+} r^d F(\mathrm{d}r)\,\mu(\mathbb{R}^d).$$

We are interested in the variations of $M(\mu)$ at a microscopic or macroscopic level. To do so, we swell, resp. shrink, the volume of the balls replacing the radius r of a ball by ρr and taking the limit $\rho \to +\infty$, resp. $\rho \to 0$. In this procedure, the law of the radius is replaced by $F_\rho(\mathrm{d}r) = f(r/\rho)\mathrm{d}r/\rho$, the image measure of $F(\mathrm{d}r)$ by the change of scale $r \mapsto \rho r$. In order to derive non-trivial asymptotics, the intensity λ of the balls is changed accordingly and we shall write $\lambda(\rho)$ to underline that from now on the intensity depends on the scaling parameter ρ. In what follows, we are thus interested in the following random field:

$$M_\rho(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} m\mu(B(x,r))N_{\lambda(\rho),\rho}(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m)$$

where $N_{\lambda(\rho),\rho}(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m)$ is the Poisson measure with intensity $\lambda(\rho)\mathrm{d}x F_\rho(\mathrm{d}r)G(\mathrm{d}m)$. The limit $\rho \to 0$ is interpreted as zoom-out in the random configurations of balls and this is relevant when the behavior of f is known at $+\infty$, i.e. $\epsilon = -1$ in (2). In this case, we investigate the macroscopic variations of M. On the contrary, $\rho \to +\infty$ is interpreted as zoom-in in space and this is relevant when the behavior of f is known at 0, i.e. $\epsilon = +1$ in (2) and this is the microscopic variations that are investigated.

Remark 1.1. As observed before, the choice $G = \delta_1$ recovers the setting of [2] for non-weighted random balls, see (4) therein. If $d = 1$, a *verbatim* replacement of $B(x,r) = (x-r, x+r)$ by $(x, x+r)$ and the choice $\mu = |\cdot \cap (0,t)|$ recover the field studied in [4] in the "continuous flow reward model", see (18) therein.

2. Results

We exhibit normalization terms $n(\rho)$ such that the normalized centered random field $n(\rho)^{-1}\big(M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]\big)$ converges in finite-dimensional distribution (*f.d.d.*) to a limit random field. In what follows, we are interested in *f.d.d.* convergence on subspaces $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ and we will denote it by $\xrightarrow{\widetilde{\mathcal{M}}}$.

It is natural to investigate first the behavior of the random field giving the density of the weight at each point which in our notations rewrites $(M_\rho(\delta_y))_{y\in\mathbb{R}^d}$. The heuristic is the following. The average numbers of balls containing the point y is given by

$$\mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} \mathbf{1}_{\{y\in B(x,r)\}}N_{\lambda(\rho),\rho}(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m)\right] = V\lambda(\rho)\rho^d,$$

where $V = c_d\int r^d F(\mathrm{d}r)$ is the expected volume of a random ball and c_d stands for the volume of the Euclidean unit ball in $\mathbb{R}^d$. Since the weights belong to the domain of attraction of an α-

stable distribution, it is natural to introduce the scaling $n_0(\rho) = \lambda(\rho)^{1/\alpha}\rho^{d/\alpha}$. Convergence of the normalized and centered random variable $M_\rho(\delta_y)$ to an α-stable distribution is obtained if we suppose that $\lambda(\rho)\rho^d \to +\infty$ when $\rho \to 0^{-\varepsilon}$. Heuristically, the dependence between $M_\rho(\delta_{y_1})$ and $M_\rho(\delta_{y_2})$ is given by the weights of the balls containing both points y_1 and y_2. In the zoom-in case ($\varepsilon = -1$, $\rho \to +\infty$), the balls are very large yielding total dependence at the limit and we have:

$$n_0(\rho)^{-1}(X_\rho(\delta_y) - \mathbb{E}[X_\rho(\delta_y)]) \xrightarrow{f.d.d.} W_\alpha, \quad y \in \mathbb{R}^d \tag{5}$$

where $W_\alpha(y) \equiv W_\alpha$ is a constant random field distributed according to $S_\alpha(\sigma V^{1/\alpha}, b, 0)$. In the zoom-out case ($\varepsilon = -1$, $\rho \to 0$), the balls are very small yielding independence at the limit and we have:

$$n_0(\rho)^{-1}(M_\rho(\delta_y) - \mathbb{E}[M_\rho(\delta_y)]) \xrightarrow{f.d.d.} W_\alpha(\delta_y), \quad y \in \mathbb{R}^d, \tag{6}$$

where $W_\alpha(\delta_y)$, $y \in \mathbb{R}^d$, are i.i.d. $S_\alpha(\sigma V^{1/\alpha}, b, 0)$ distributed. Similar results as in (5) and in (6) hold true for f.d.d. convergence on the space of measures with finite support. Since these results are not surprising, their proofs are omitted and in what follows we investigate convergence for more general measures.

2.1. Preliminaries on measured spaces

We introduce a subspace $\mathcal{M}_{\alpha,\beta} \subset \mathcal{M}$ on which we will show the convergence of the rescaled generalized random field $M_\rho(\mu)$.

Definition 2.1. For $1 < \alpha \le 2$ and $\beta > 0$, let $\mathcal{M}_{\alpha,\beta}$ be the subset of measures $\mu \in \mathcal{M}$ satisfying for some finite constant C and some $0 < p < \beta < q$:

$$\gamma(r) := \int_{\mathbb{R}^d} |\mu(B(x,r))|^\alpha \mathrm{d}x \le C(r^p \wedge r^q) \tag{7}$$

where for reals a, b: $a \wedge b = \min(a, b)$.

Here and in what follows, C is a finite constant that may change at each occurrence. Some elementary properties of the spaces $\mathcal{M}_{\alpha,\beta}$ are given in the following proposition.

Proposition 2.2.

(i) *$\mathcal{M}_{\alpha,\beta}$ is a linear subspace of $\mathcal{M}$ on which*

$$\forall \mu \in \mathcal{M}_{\alpha,\beta}, \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x,r))|^\alpha r^{-\beta-1} \mathrm{d}x\mathrm{d}r < +\infty.$$

(ii) *$\mathcal{M}_{\alpha,\beta}$ is closed under translations, rotations and dilatations, i.e. when $\mu \in \mathcal{M}_{\alpha,\beta}$, $\tau_s\mu$, $\Theta\mu$ and μ_a are also in $\mathcal{M}_{\alpha,\beta}$ where for any Borelian set A and for $s \in \mathbb{R}^d$, $\Theta \in \mathcal{O}(\mathbb{R}^d)$, $a \in \mathbb{R}_+$*

$$\tau_s\mu(A) = \mu(A - s), \qquad \Theta\mu(A) = \mu(\Theta^{-1}A), \qquad \mu_a(A) = \mu(a^{-1}A).$$

(iii) *When $\alpha \le \alpha'$, we have $\mathcal{M}_{\alpha,\beta} \subset \mathcal{M}_{\alpha',\beta}$.*

(iv) *When $\beta \ge d$, the space $\mathcal{M}_{\alpha,\beta}$ is included in the subspace of diffuse measures (i.e. such that $\mu(\{x\}) = 0$ for any $x \in \mathbb{R}^d$).*

(v) *When $\beta \le d$, the space $\mathcal{M}_{\alpha,\beta}$ is included in the subspace of centered measures (i.e. such that $\mu(\mathbb{R}^d) = 0$).*

Observe that Dirac measures δ_y, $y \in \mathbb{R}^d$, are not in $\mathcal{M}_{\alpha,\beta}$. However, explicit examples of measure in $\mathcal{M}_{\alpha,\beta}$ are given in the following proposition. Absolutely continuous measures (with respect to the Lebesgue measure) $\mu(\mathrm{d}x) = \phi(x)\mathrm{d}x$ with integrable density $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$ will play an important role. In this case, we shall (abusively) note $\mu \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$.

Proposition 2.3.

(i) *If $d < \beta < \alpha d$, any measure $\mu \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$ belongs to $\mathcal{M}_{\alpha,\beta}$.*
(ii) *If $d - 1 < \beta < d$, any centered measure $\mu(\mathrm{d}x) = \phi(x)\mathrm{d}x \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \|y\| |\phi(y)| \mathrm{d}y < +\infty$ belongs to $\mathcal{M}_{\alpha,\beta}$, as well as any centered measure with finite support.*

Note that in particular, when $d < \beta < \alpha d$ (resp. $d - 1 < \beta < d$), $\mathcal{M}_{\alpha,\beta}$ contains the space $\mathcal{S}$ of measures with density in the Schwartz class (resp. $\mathcal{S}_0$ the space of centered measures with density in the Schwartz class). Note also that when $\alpha = 2$, the conditions supposed in [2] on the measure μ (expressed in terms of Riesz energy) imply that $\mu \in \mathcal{M}_{2,\beta}$. By analogy with the case $\alpha = 2$, we suspect the space $\mathcal{M}_{\alpha,\beta}$ to be reduced to $\{0\}$ whenever $\beta \le d - 1$ or $\beta \ge \alpha d$, but we have no formal proof of these facts. However, we refer to Theorem 2.19 for a positive result when $\beta > \alpha d$.

2.2. Limit theorems for the rescaled weighted random ball model

We now come to the main results of this paper, *viz.* limit theorems for the rescaled generalized random fields M_ρ and for configurations $\mu \in \mathcal{M}_{\alpha,\beta}$ on the balls. As in [7] and [4] (for $\epsilon = -1$), several regimes appear according to the density of large/small balls in the limit. More precisely, using (2):

Zoom-out case. ($\epsilon = -1$, i.e. $\beta > d$ and $\rho \to 0$). The mean number of balls with radius larger than one that cover the origin is given by

$$\int_{\mathbb{R}^d \times \mathbb{R}+} \mathbf{1}_{\|x\|<r} \mathbf{1}_{r>1} \lambda(\rho) \mathrm{d}x F_\rho(\mathrm{d}r) = c_d \lambda(\rho) \int_1^{+\infty} r^{\mathrm{d}} F_\rho(\mathrm{d}r) \sim_{\rho \to 0} \frac{c_d C_\beta}{\beta - d} \lambda(\rho) \rho^\beta.$$

Consequently, we distinguish the following three scaling regimes:

- large-balls scaling: $\lambda(\rho)\rho^\beta \to +\infty$,
- intermediate scaling: $\lambda(\rho)\rho^\beta \to a \in (0, +\infty)$,
- small-balls scaling: $\lambda(\rho)\rho^\beta \to 0$.

Zoom-in case. ($\epsilon = +1$, i.e. $\beta < d$ and $\rho \to +\infty$). The mean number of balls with radius less than one that cover the origin is given by

$$\int_{\mathbb{R}^d \times \mathbb{R}+} \mathbf{1}_{\|x\|<r} \mathbf{1}_{r<1} \lambda(\rho) \mathrm{d}x F_\rho(\mathrm{d}r) = c_d \lambda(\rho) \int_0^{1} r^{\mathrm{d}} F_\rho(\mathrm{d}r) \sim_{\rho \to +\infty} \frac{c_d C_\beta}{d - \beta} \lambda(\rho) \rho^\beta.$$

In this case, the three scaling regimes are:

- small-balls scaling: $\lambda(\rho)\rho^\beta \to +\infty$,
- intermediate scaling: $\lambda(\rho)\rho^\beta \to a \in (0, +\infty)$,
- large-balls scaling: $\lambda(\rho)\rho^\beta \to 0$.

In what follows, we study precisely the limiting shape of the random balls by investigating the fluctuations of $M(\mu)$ around its mean. Three different limit fields are exhibited according to the scaling performed.

2.2.1. Stable regime with dependence

In this section, we investigate the behavior of M under the scaling $\rho^\beta\lambda(\rho) \to +\infty$. In this case the limiting field is given by an α-stable integral. We recall that the stable stochastic integral of f with respect to an α-stable random measure with control measure m is well defined whenever $f \in L^\alpha(\mathrm{d}m)$ and in this case, this stochastic integral follows an α-stable distribution. We refer to [10] for a complete account on stable measures and integrals. The asymptotic of the rescaled generalized fields M_ρ is given by the following result:

Theorem 2.4. *Suppose $\rho^\beta\lambda(\rho) \to +\infty$ when $\rho \to 0^{-\epsilon}$. Let $n_1(\rho) = \lambda(\rho)^{1/\alpha}\rho^{\beta/\alpha}$. We have*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n_1(\rho)} \xrightarrow{\mathcal{M}_{\alpha,\beta}} Z_\alpha(\cdot) \quad \rho \to 0^{-\epsilon} \tag{8}$$

where $Z_\alpha(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r))M_\alpha(\mathrm{d}r,\mathrm{d}x)$ is a stable integral with respect to the α-stable measure M_α with control measure $\sigma^\alpha C_\beta r^{-1-\beta}\mathrm{d}r\mathrm{d}x$ and constant skewness function b given in the domain of attraction of G.

Note that $Z_\alpha(\mu)$ makes sense as soon as $\int_{\mathbb{R}\times\mathbb{R}^+} |\mu(B(x,r))|^\alpha r^{-1-\beta}\mathrm{d}r\mathrm{d}x < +\infty$ (see Proposition 2.2-(i)). However, we need the stronger assumption $\mu \in \mathcal{M}_{\alpha,\beta}$ in order to derive (8). Roughly speaking, the control (7) of $\mu \in \mathcal{M}_{\alpha,\beta}$ allows to replace F by its tails behavior given in (2) in asymptotic estimate.

Due to the invariance by translation and rotation of the Lebesgue measure, the self-similarity of stable integral and the (global) invariance by rotation of the balls and because of Proposition 2.2-(ii), we derive the following properties for the limit field Z_α of Theorem 2.4:

Proposition 2.5.

(i) *The field Z_α is stationary on $\mathcal{M}_{\alpha,\beta}$, that is:*

$$\forall\mu \in \mathcal{M}_{\alpha,\beta}, \forall s \in \mathbb{R}^d, \qquad Z_\alpha(\tau_s\mu) \stackrel{f.d.d.}{=} Z_\alpha(\mu).$$

(ii) *The field Z_α is isotropic on $\mathcal{M}_{\alpha,\beta}$, that is:*

$$\forall\mu \in \mathcal{M}_{\alpha,\beta}, \forall\Theta \in \mathcal{O}(\mathbb{R}^d), \qquad Z_\alpha(\Theta\mu) \stackrel{f.d.d.}{=} Z_\alpha(\mu).$$

(iii) *The field Z_α is self-similar on $\mathcal{M}_{\alpha,\beta}$ with index $(d-\beta)/\alpha$, that is:*

$$\forall\mu \in \mathcal{M}_{\alpha,\beta}, \forall a > 0, \qquad Z_\alpha(\mu_a) \stackrel{f.d.d.}{=} a^{(d-\beta)/\alpha} Z_\alpha(\mu).$$

Remark 2.6. The covariation gives an insight into the structure of the spatial dependence of the stable generalized field. It is a generalization of the usual notion of covariance to the stable framework. Here, for $\mu_1, \mu_2 \in \mathcal{M}_{\alpha,\beta}$, the covariation of $Z_\alpha(\mu_1)$ on $Z_\alpha(\mu_2)$ is given by

$$[Z_\alpha(\mu_1), Z_\alpha(\mu_2)]_\alpha = \sigma^\alpha C_\beta \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu_1(B(x,r))\epsilon(\mu_2(B(x,r)))|\mu_2(B(x,r))|^{\alpha-1} r^{-\beta-1}\mathrm{d}r\mathrm{d}x.$$

Note that the integral above is well defined by Hölder's inequality since μ_1 and μ_2 belong to $\mathcal{M}_{\alpha,\beta}$. We refer to [10] for a definition and properties of the covariation. Note that unlike the Gaussian case, the covariation structure is not sufficient to characterize the distribution of the generalized random field. However, since even if μ_1 and μ_2 have disjoint supports,

$[Z_\alpha(\mu_1), Z_\alpha(\mu_2)]_\alpha \neq 0$, $Z_\alpha(\mu_1)$ and $Z_\alpha(\mu_2)$ are not independent and the random field Z_α is stable with dependence.

Remark 2.7. Note that when $d-1<\beta<d$, $\mu_z=\delta_z-\delta_0$ for $z\in\mathbb{R}^d$ belongs to $\mathcal{M}_{\alpha,\beta}$. For such a measure, when moreover $b=0$ (i.e. when G in our model is symmetric), our limiting field rewrites

$$Z_\alpha(\mu_z)=\int_{\mathbb{R}^d\times\mathbb{R}^+}\mathbf{1}_{B(z,r)\Delta B(0,r)}M_\alpha(\mathrm{d}x,\mathrm{d}r)$$

where $A\Delta B=(A\setminus B)\cup(B\setminus A)$. In this case, we recover the so-called (α,H)-*Takenaka field* with $H=(d-\beta)/\alpha$. It is self-similar with index H, with stationary increments and almost surely with continuous sample paths, see [2, p. 25] or [10, Sect. 8.4].

Remark 2.8. When $d=1$, $\beta\in(1,\alpha)$ and $\mu_t=|\cdot\cap(0,t)|$, the field $Z_\alpha(\mu_t)$ coincides with the *Telecom process* obtained in the fast connection rate for the "continuous flow reward model" in [4, Th. 2], see also Remark 1.1 above. Moreover for $\alpha=2$, $Z_2(\mu_t)$ is a fractional Brownian motion of Hurst index $H=(3-\beta)/2\in(1/2,1)$ (note that, for $a>0$, $\mu_{at}(A)=a\mu_t(a^{-1}A)$).

Remark 2.9. When $\alpha=2$, Theorem 2.4 exhibits a Gaussian limit field and generalizes Theorem 2.1 in [2] with random weights. Indeed, in this case, we have (up to some multiplicative constant) $Z_2=W_\beta$.

Remark 2.10. A natural complementary result to be investigated is the tightness of M_ρ after normalization and centering which would allow to turn f.d.d. convergences into weak functional convergences. In dimension $d=1$, only partial tightness results are available for the processes studied in [4,9] (see Section 4 on "continuous flow reward model" in [4] and the remarks of Th. 1, Th. 2 and Th. 3 in [9]). In the case of generalized random fields, tightness issue is more difficult to tackle due to the lack of tractable tightness criterion.

2.2.2. *Poissonian regime*

In this section, we investigate the behavior of M under the scaling $\rho^\beta\lambda(\rho)\to a\in(0,\infty)$. In this case, the limiting field is given by a compensated Poisson integral and we refer to [8] for a general description of Poisson integral. We have:

Theorem 2.11. *Suppose $\lambda(\rho)\rho^\beta\to a^{d-\beta}$ when $\rho\to 0^{-\epsilon}$ for some $a>0$. We have*

$$M_\rho(\mu)-\mathbb{E}[M_\rho(\mu)]\xrightarrow{\mathcal{M}_{\alpha,\beta}} J(\mu_a),\quad \rho\to 0^{-\epsilon}$$

where μ_a is the dilatation of μ and J is the compensated Poisson integral

$$J(\mu)=\int_{\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^+} m\mu(B(x,r))\widetilde{N}_\beta(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m) \tag{9}$$

with respect to the compensated Poisson random measure $\widetilde{N}_\beta$ with intensity given by $C_\beta r^{-\beta-1}\mathrm{d}x\mathrm{d}rG(\mathrm{d}m)$.

Note that the Poisson integral in (9) above is well defined since

$$\int_{\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^+}\left(|m\mu(B(x,r))|\wedge(m\mu(B(x,r)))^2\right)r^{-\beta-1}\mathrm{d}x\mathrm{d}rG(\mathrm{d}m)<+\infty \tag{10}$$

see Section 3.4. As the stable field Z_α, the Poisson field J enjoys similar properties. However, note that in contrast to Z_α, J is not self-similar but (and similarly to [2], see also [5]) J satisfies an aggregate similarity property.

Proposition 2.12. *The field J is stationary and isotropic on $\mathcal{M}_{\alpha,\beta}$. Moreover, J is aggregate similar, viz. $\forall \mu \in \mathcal{M}_{\alpha,\beta}, \forall m \geq 1$,*

$$J(\mu_{a_m}) \stackrel{f.d.d.}{=} \sum_{i=1}^{m} J^i(\mu) \tag{11}$$

where J^i, $1 \leq i \leq m$, are independent copies of J and $a_m = m^{1/(d-\beta)}$.

The proof of this proposition follows from straightforward computation and will be omitted. A comparison of the limiting procedures in Theorem 2.4 where $\lambda(\rho)\rho^\beta \to +\infty$ and in Theorem 2.11 where $\lambda(\rho)\rho^\beta \to a^{d-\beta}$ suggests that when $a^{d-\beta} \to +\infty$, we can recover Z_α from J. This is true and precisely stated in the following proposition:

Proposition 2.13. *When $a^{d-\beta} \to +\infty$, we have $\frac{1}{a^{(d-\beta)/\alpha}} J(\mu_a) \xrightarrow{\mathcal{M}_{\alpha,\beta}} Z_\alpha(\mu)$.*

Remark 2.14. As in Remark 2.8, when $d = 1$ and $\mu_t = |\cdot \cap (0, t)|$, the field $J(\mu_t)$ coincides with the *intermediate Telecom process* obtained in the intermediate connection rate for the "continuous flow reward model" in [4, Th. 1], see also Remark 1.1 above.

Remark 2.15. When $\alpha = 2$, Theorem 2.11 generalizes Theorem 2.5 in [2] with random weights. The field J recovers J_β in [2] when the random weights in our model are constant. Otherwise the law of J depends on the law G of the weight.

2.2.3. Stable regime with independence for small radius

In this section, we investigate the behavior of M under the scaling $\rho^\beta \lambda(\rho) \to 0$, but we restrict to the case $d < \beta < \alpha d$, i.e. $\epsilon = -1$ and $\rho \to 0$, see Section 2.2.4 for $\beta > \alpha d$. The case $m \equiv 1$ is considered in Theorem 2 (iii) of [7] and we extend here the results and proofs to the case when the weights are random and belong to the normal domain of attraction of a stable distribution. In comparison to the case $\beta < d$, the tails of the law of the radius are lighter and thus the radius considered are small. We show that the asymptotic behavior is given again by a stable field but with index $\gamma = \beta/d$ and defined on $\mathbb{R}^d$. Moreover in contrast to the stable field Z_α of Section 2.2.1, this new field exhibits independence.

Theorem 2.16. *Let $d < \beta < \alpha d$ and suppose that $\lambda(\rho) \to +\infty$ and $\lambda(\rho)\rho^\beta \to 0$ as $\rho \to 0$. Then with $n_2(\rho) := \lambda(\rho)^{d/\beta} \rho^d$ and $\gamma = \beta/d \in (1, \alpha)$, we have*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n_2(\rho)} \xrightarrow{L^1(\mathbb{R}^d)\cap L^\alpha(\mathbb{R}^d)} \widetilde{Z}_\gamma(\cdot)$$

where, for $\mu(\mathrm{d}x) = \phi(x)\mathrm{d}x$, $\widetilde{Z}_\gamma(\mu) = \int_{\mathbb{R}^d} \phi(x)\widetilde{M}_\gamma(\mathrm{d}x)$ is a stable integral with respect to the γ-stable measure M_γ with control measure $\sigma_\gamma^\gamma\,\mathrm{d}x$ for

$$\sigma_\gamma^\gamma = \frac{c_d^\gamma C_\beta}{\mathrm{d}} \int_{\mathbb{R}_+} \frac{1-\cos(r)}{r^{1+\gamma}} \mathrm{d}r \int_{\mathbb{R}} |m|^\gamma G(\mathrm{d}m)$$

and with constant skewness function equals to

$$b_\gamma = -\frac{\int_{\mathbb{R}} \varepsilon(m)|m|^\gamma G(\mathrm{d}m)}{\int_{\mathbb{R}} |m|^\gamma G(\mathrm{d}m)}. \tag{12}$$

Note that the integrals above are well defined when $d < \beta < \alpha d$ (see Lemma 3.1). The limiting field $\widetilde{Z}_\gamma$ enjoys similar properties as Z_α and J:

Proposition 2.17. *The field $\widetilde{Z}_\gamma$ is stationary, isotropic and self-similar with index $(1-\gamma)d/\gamma$.*

Remark 2.18. As in Remarks 2.8 and 2.14, when $d = 1$ and $\phi_t = \mathbf{1}_{(0,t)}$, the field $\widetilde{Z}_\gamma(\phi_t)$ coincides with the process obtained in the slow connection rate for the "continuous flow reward model" in [4, Th. 3], see also Remark 1.1. In this particular case, $\widetilde{Z}_\gamma(\phi_t)$ is a γ-stable Lévy process.

2.2.4. Stable regime with independence for very small radius

When the tails of the radii are lighter than that in Section 2.2.3, i.e. $\beta > \alpha d$, the same stable regime with independence as in Section 2.2.3 appears but under a different normalization $n_3(\rho) := \lambda(\rho)^{1/\alpha}\rho^d$ and a different stability index α. As previously, since $\beta > \alpha d$, we have $\epsilon = -1$ and the limits are taken when $\rho \to 0$, i.e. the limiting scheme is a zooming-out procedure.

Theorem 2.19. *Let $\beta > \alpha d$ and suppose that $\lambda(\rho) \to +\infty$ as $\rho \to 0$. Let $n_3(\rho) := \lambda(\rho)^{1/\alpha}\rho^d$, then*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n_3(\rho)} \xrightarrow{L^1(\mathbb{R}^d)\cap L^\alpha(\mathbb{R}^d)} \widetilde{Z}_\alpha(\cdot)$$

where, for $\mu(\mathrm{d}x) = \phi(x)\mathrm{d}x$, $\widetilde{Z}_\alpha(\mu) = \int_{\mathbb{R}^d} \phi(x)\widetilde{M}_\alpha(\mathrm{d}x)$ is a stable integral with respect to the α-stable measure M_α with control measure $\sigma_\alpha \mathrm{d}x$ with $\sigma_\alpha = \sigma c_d \left(\int_{\mathbb{R}^+} r^{\alpha d} F(\mathrm{d}r)\right)^{1/\alpha}$ and constant skewness equal to b.

Remark 2.20. It is worth noting that in both Theorems 2.16 and 2.19, the stable regime is driven by the parameter $\gamma = (\beta/d)\wedge\alpha$, since the normalization is $\lambda(\rho)^{1/\gamma}\rho^d$ and the stability index is γ.

Actually, only the asymptotics of the law with the heavier tails contribute to the limit while the law with the lighter tails appears only (but globally) as a mere parameter in the limit. In particular, observe that Theorem 2.19 applies for any distribution F such that $\int_{\mathbb{R}^+} r^{\alpha d} F(\mathrm{d}r) < +\infty$.

Remark 2.21. When $d = 1$ and $\mu_t = |\cdot \cap (0,t)|$, we recover (ii) in Theorem 4 of [4].

3. Proof of the results

In what follows, note that the linearity of the random functionals M_ρ and of the stochastic integrals in W_α, $\widetilde{W}_\alpha$, Z_α, J and $\widetilde{Z}_\gamma$, together with the Cramér—Wold device imply that the convergence of the finite-dimensional distributions of the centered and renormalized version of M_ρ is equivalent to the convergence of the one-dimensional distributions. To do so, we will explicitly compute the limits of the characteristic functions, denoting φ_X for the characteristic function of a random variable X. Observe that the characteristic function of $n(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)])$ rewrites:

$$\varphi_{n(\rho)^{-1}(M_\rho(\mu)-\mathbb{E}[M_\rho(\mu)])}(\theta) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} \Psi\big(n(\rho)^{-1}\theta m\mu(B(x,r))\big)\lambda(\rho)\mathrm{d}x F_\rho(\mathrm{d}r)G(\mathrm{d}m)\right)$$

where $\Psi(u) = \mathrm{e}^{\mathrm{i}u} - 1 - \mathrm{i}u$, see [8]. Integrating first with respect to the probability $G(\mathrm{d}m)$, we have

$$\varphi_{n(\rho)^{-1}(M_\rho(\mu)-\mathbb{E}[M_\rho(\mu)])}(\theta) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} \lambda(\rho)\,\Psi_G\big(n(\rho)^{-1}\theta\mu(B(x,r))\big)\mathrm{d}x F_\rho(\mathrm{d}r)\right) \tag{13}$$

where $\Psi_G(u) = \int_{\mathbb{R}} \Psi(mu)G(\mathrm{d}m)$. We also recall that the characteristic function of the stable distribution $S_\alpha(\sigma, b, \tau)$ is given by $\exp(-\sigma^\alpha|x|^\alpha(1 - \mathrm{i}b\varepsilon(\theta)\tan(\pi\alpha/2)) + \mathrm{i}\tau\theta)$.

3.1. Preliminary lemmas

In this section, we collect some useful lemmas that will be needed in the proof of our limit Theorems 2.4, 2.11 and 2.16. We recall the following estimate for the characteristic function of distribution in the domain of attraction of a stable law:

Lemma 3.1. *Suppose X is in the domain of attraction of an α-stable law $S_\alpha(\sigma, b, 0)$ for some $\alpha > 1$. Then*

$$\varphi_X(\theta) - 1 - \mathrm{i}\theta\mathbb{E}[X] \sim_0 -\sigma^\alpha|\theta|^\alpha(1 - \mathrm{i}\varepsilon(\theta)\tan(\pi\alpha/2)b).$$

Furthermore, there is some $C > 0$ such that for any $\theta \in \mathbb{R}$,

$$|\varphi_X(\theta) - 1 - \mathrm{i}\theta\mathbb{E}[X]| \le C|\theta|^\alpha.$$

The following lemma is a reformulation from Lemma 2.4 in [2]. It shows that in the scaling limit $\rho \to 0^{-\epsilon}$, the behavior of F_ρ is given by the power tail of F. This is crucial in several estimates.

Lemma 3.2. *Let F be as in (2) and $\epsilon = \pm 1$. Assume that g is a continuous function on $\mathbb{R}^+$ such that for some $0 < p < \beta < q$, there exists some $C > 0$ such that*

$$|g(r)| \le C(r^p \wedge r^q). \tag{14}$$

Assume furthermore that $(g_\rho)_{\rho>0}$ is a family of continuous functions such that

$$\lim_{\rho\to 0^{-\epsilon}} |g(r) - g_\rho(r)| = 0 \quad \text{and} \quad |g(r) - g_\rho(r)| \le C(r^p \wedge r^q). \tag{15}$$

Then

$$\int_{\mathbb{R}^+} g_\rho(r)F_\rho(\mathrm{d}r) \sim C_\beta\rho^\beta \int_{\mathbb{R}^+} g(r)r^{-1-\beta}\mathrm{d}r \quad \textit{when } \rho \to 0^{-\epsilon}.$$

In the proof of Theorem 2.4 and of Theorem 2.11, this lemma will be used in the particular case where $g_\rho = g$ and g satisfies condition (14). Roughly speaking, the proof of Lemma 3.2 consists in taking the limit in the integral. This is authorized by the dominated convergence theorem under (14) and (15). We refer to [2] for more details.

3.2. Proofs of Propositions 2.2 and 2.3

Proof of Proposition 2.2. Proof of (i). If (7) holds true for μ_1 with $p_1 < \beta < q_1$ and for μ_2 with $p_2 < \beta < q_2$, then (7) holds true for μ_1 and μ_2 with $p = p_1 \vee p_2 < \beta$ and $q = q_1 \wedge q_2 > \beta$ (possibly with a different constant C). For all $a_1, a_2 \in \mathbb{R}$:

$$\begin{aligned}\int_{\mathbb{R}^d} |(a_1\mu_1 + a_2\mu_2)(B(x,r))|^\alpha \mathrm{d}x &= \|(a_1\mu_1 + a_2\mu_2)(B(x,r))\|_\alpha^\alpha \\ &\le \left(|a_1|\|\mu_1(B(x,r))\|_\alpha + |a_2|\|\mu_2(B(x,r))\|_\alpha\right)^\alpha \\ &\le \left((|a_1|^\alpha C(r^p \wedge r^q))^{1/\alpha} + (|a_2|^\alpha C(r^p \wedge r^q))^{1/\alpha}\right)^\alpha \\ &= C(|a_1| + |a_2|)^\alpha (r^p \wedge r^q).\end{aligned}$$

This is (7) for $a_1\mu_1 + a_2\mu_2$.

Proof of (ii). Since $(\tau_s\mu)(B(x,r)) = \mu(B(x-s,r))$, $(\theta\mu)(B(x,r)) = \mu(B(\Theta^{-1}x,r))$, $\mu_a(B(x,r)) = \mu(B(a^{-1}x, a^{-1}r))$, the closeness of $\mathcal{M}_{\alpha,\beta}$ by translations τ_s, by rotations Θ and by dilatations $x \mapsto ax$ follow straightforwardly from the invariance of the Lebesgue measure by translations, by rotation, and by an immediate change of variable in (7).

Proof of (iii). Since $|\mu|(\mathbb{R}^d) < +\infty$, for $\mu \in \mathcal{M}_{\alpha,\beta}$ and $\alpha \le \alpha'$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha'} \mathrm{d}x &= \int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha'-\alpha} |\mu(B(x,r))|^\alpha \mathrm{d}x \\ &\le |\mu|(\mathbb{R}^d)^{\alpha'-\alpha} \int_{\mathbb{R}^d} |\mu(B(x,r))|^\alpha \mathrm{d}x \\ &\le C(r^p \wedge r^q)\end{aligned}$$

which proves $\mu \in \mathcal{M}_{\alpha',\beta}$.

Proof of (iv). We prove that $\mu \in \mathcal{M}_{\alpha,\beta}$ is diffuse when $\beta > d$. Indeed, suppose that μ has an atom a, then for small enough r, $\gamma(r) \ge |\mu(a)/2|^\alpha c_d r^d$, where we recall that $\gamma(r)$ is defined in (7). To see this, let $\varepsilon > 0$ be such that $||\mu|(B(a,\varepsilon)) - |\mu(a)|| < |\mu(a)|/2$. Then, for every $r < \varepsilon/2$ and $x \in B(a,r)$, $|\mu(B(x,r))| \ge |\mu(a)|/2$. Integrating on $x \in B(a,r)$, we get $\gamma(r) \ge (|\mu(a)|/2)^\alpha c_d r^d$. This is in contradiction with (7) which rewrites $\gamma(r) \le Cr^q$ for $q > \beta > d$ when r is small.

Proof of (v). We prove that $\mu \in \mathcal{M}_{\alpha,\beta}$ is centered when $\beta \le d$. We will show that

$$\gamma(r) \ge |\mu(\mathbb{R}^d)/3|^\alpha c_\mathrm{d} r^d \tag{16}$$

when r is large enough. This is sufficient to prove (v) since (7) rewrites $\gamma(r) \le Cr^p$ for $p < \beta < d$ when $r \ge 1$ which is in contradiction with (16) when $\mu(\mathbb{R}^d) \ne 0$.

The bound (16) is obvious if $\mu(\mathbb{R}^d) = 0$ and if $\mu(\mathbb{R}^d) \ne 0$, let M be such that $|\mu|(B(0,M)^c) \le |\mu(\mathbb{R}^d)|/3$. Then, for $r \ge M$ and any $x \in B(0, r-M)$, $B(0,M) \subset B(x,r)$ and $|\mu(B(x,r))| \ge |\mu(\mathbb{R}^d)| - |\mu|(B(x,r)^c) \ge 2|\mu(\mathbb{R}^d)|/3$. Integrating on $x \in B(0, r-M)$, we obtain $\gamma(r) \ge (2|\mu(\mathbb{R}^d)|/3)^\alpha c_d (r-M)^d$. This implies (16).

Proof of Proposition 2.3. Proof of (i). First, when $d < \beta < \alpha d$ and $\mu(\mathrm{d}x) = \phi(x)\mathrm{d}x \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$, we have:

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mu(B(x,r))|^\alpha \mathrm{d}x &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathbf{1}_{\{\|x-y\|<r\}} \phi(y) \mathrm{d}y \right|^\alpha \mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbf{1}_{\{\|x-y\|<r\}} |\phi(y)|^\alpha \mathrm{d}y \right) \left(\int_{\mathbb{R}^d} \mathbf{1}_{\{\|x-y\|<r\}} \mathrm{d}y \right)^{\alpha-1} \mathrm{d}x \\
&= (c_{\mathrm{d}} r^d)^{\alpha-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{\{\|x-y\|<r\}} |\phi(y)|^\alpha \mathrm{d}y \mathrm{d}x \\
&= (c_{\mathrm{d}} r^d)^{\alpha-1} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbf{1}_{\{\|x-y\|<r\}} \mathrm{d}x \right) |\phi(y)|^\alpha \mathrm{d}y \\
&= (c_{\mathrm{d}} r^d)^{\alpha} \int_{\mathbb{R}^d} |\phi(y)|^\alpha \mathrm{d}y
\end{aligned}$$

where we applied Hölder's inequality with $\alpha > 1$. Next, in the same way,

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mu(B(x,r))|^\alpha \mathrm{d}x &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathbf{1}_{\{\|x-y\|<r\}} \phi(y) \mathrm{d}y \right|^\alpha \mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbf{1}_{\{\|x-y\|<r\}} |\phi(y)| \mathrm{d}y \right) \left(\int_{\mathbb{R}^d} |\phi(y)| \mathrm{d}y \right)^{\alpha-1} \mathrm{d}x \\
&= c_{\mathrm{d}} r^d \left(\int_{\mathbb{R}^d} |\phi(y)| \mathrm{d}y \right)^{\alpha}. \qquad (17)
\end{aligned}$$

As a consequence, condition (7) holds with $p = d < \beta < q = \alpha d$, and $\mu \in \mathcal{M}_{\alpha,\beta}$.

Proof of (ii). Suppose $d - 1 < \beta < d$ and $\mu(\mathrm{d}x) = \phi(x)\mathrm{d}x \in L^1(\mathbb{R}^d)$ is centered. Using $\mu(\mathbb{R}^d) = 0$, we have:

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mu(B(x,r))|^\alpha \mathrm{d}x &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (\mathbf{1}_{\{\|x-y\|<r\}} - \mathbf{1}_{\{\|x\|<r\}}) \phi(y) \mathrm{d}y \right|^\alpha \mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathbf{1}_{\{\|x-y\|<r\}} - \mathbf{1}_{\{\|x\|<r\}}|^\alpha |\phi(y)| \mathrm{d}y \right) \\
&\quad \times \left(\int_{\mathbb{R}^d} |\phi(y)| \mathrm{d}y \right)^{\alpha-1} \mathrm{d}x. \qquad (18)
\end{aligned}$$

Let $\theta(z) = |B(0,1) \Delta B(z,1)|$ denotes the volume of the symmetric difference of the balls with unit radius centered at 0 and at $z \in \mathbb{R}^d$. We have,

$$\int_{x \in \mathbb{R}^d} |\mathbf{1}_{\{\|x-y\|<r\}} - \mathbf{1}_{\{\|x\|<r\}}| \mathrm{d}x = r^d \theta\left(\frac{y}{r}\right).$$

The function θ is continuous, upper bounded by c_d and $\theta(z) = O(\|z\|)$ as $z \to 0$. As a consequence, the global estimate $|\theta(z)| \le C\|z\|$ holds true for some $C > 0$. This entails

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mu(B(x,r))|^\alpha \mathrm{d}x &= \left(\int_{\mathbb{R}^d} |\phi(y)| \mathrm{d}y \right)^{\alpha-1} \int_{\mathbb{R}^d} r^d \theta\left(\frac{y}{r}\right) \phi(y) |\mathrm{d}y \\
&\le \left(\int_{\mathbb{R}^d} |\phi(y)| \mathrm{d}y \right)^{\alpha-1} \int_{\mathbb{R}^d} C\|y\| |\phi(y)| \mathrm{d}y r^{d-1} \\
&\le C r^{d-1}. \qquad (19)
\end{aligned}$$

As a consequence, condition (7) holds true with $p = d-1 < \beta < q = d$ because (17) still holds true, and finally $\mu \in \mathcal{M}_{\alpha,\beta}$.

Alternatively, if μ has a finite support $\{a_1, \dots, a_p\}$, let $\delta > 0$ such that for $1 \le i \le p$, $B(a_i, \delta) \cap \operatorname{Supp}(\mu) = \{a_i\}$. For $r < \delta/2$,

$$\begin{aligned}\gamma(r) &= \int_{\mathbb{R}^d} |\mu(B(x,r))|^\alpha \mathrm{d}x \\ &= \sum_{i=1}^p \int_{B(a_i,r)} |\mu(B(x,r))|^\alpha \mathrm{d}x \\ &= \sum_{i=1}^p \int_{B(a_i,r)} |\mu(a_i)|^\alpha \mathrm{d}x \\ &= c_d \sum_{i=1}^p |\mu(a_i)|^\alpha r^d = O(r^d). \end{aligned} \tag{20}$$

Next, let M be such that $\mu(B(0,M)^c) = 0$ and note that $\mu(B(x,r)) = 0$ when $B(x,r) \cap B(0,M) = \emptyset$ or when $B(0,M) \subset B(x,r)$ since $\mu(\mathbb{R}^d) = 0$. We derive $\mu(B(x,r)) = 0$ when $\|x\| \le r - M$ or when $\|x\| \ge M + r$. Since μ is a finite measure, we have

$$\begin{aligned}\gamma(r) &= \int_{r-M \le \|x\| \le r+M} |\mu(B(x,r))|^\alpha \mathrm{d}x \\ &\le c_d\big((r+M)^d - (r-M)^d\big)(|\mu|(\mathbb{R}^d))^\alpha \\ &= O(r^{d-1}), \quad r \to +\infty.\end{aligned}$$

Together with (20), this yields condition (7) with $p = d-1 < \beta$ and $q = d > \beta$. □

Remark 3.3 (*On the bound for large radii*). Note that in order to derive the bound $\gamma(r) \le r^p$ for $p < \beta$ when r is large, the existence of a density for μ is not required. We can instead suppose that μ satisfy some tail condition: for some $\tilde{\eta} > d/\alpha$

$$|\mu|(B(0,R)^c) = O(R^{-\tilde{\eta}}) \quad \text{as } R \to +\infty.$$

3.3. *Proof of Theorem 2.4*

The characteristic function of the stable integral $Z_\alpha(\mu)$ is given by

$$\begin{aligned}&\varphi_{Z_\alpha(\mu)}(\theta) \\ &\quad = \exp\left(-C_\beta \sigma^\alpha \int_{\mathbb{R}^d \times \mathbb{R}^+} |\theta\mu(B(x,r))|^\alpha (1 - \mathrm{i}\varepsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b) r^{-1-\beta} \mathrm{d}r\mathrm{d}x\right).\end{aligned} \tag{21}$$

Since the characteristic function of the Poisson integral $n_1(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)])$ is given by (13), comparing (21) and (13), it is sufficient to show that

$$\begin{aligned}&\lim_{\rho \to 0^{-\epsilon}} \int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda(\rho)\Psi_G(n_1(\rho)^{-1}\theta\mu(B(x,r)))\mathrm{d}x F_\rho(\mathrm{d}r) \\ &\quad = -C_\beta\sigma^\alpha \int_{\mathbb{R}^d \times \mathbb{R}^+} |\theta\mu(B(x,r))|^\alpha (1 - \mathrm{i}\varepsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b) r^{-1-\beta}\mathrm{d}r\mathrm{d}x.\end{aligned} \tag{22}$$

Since $n_1(\rho) = (\lambda(\rho)\rho^\beta)^{1/\alpha} \to +\infty$, Lemma 3.1 applies and yields

$$\lambda(\rho)\,\Psi_G\big(n_1(\rho)^{-1}\theta\mu(B(x,r))\big) \sim -\sigma^\alpha\rho^{-\beta}|\theta|^\alpha|\mu(B(x,r))|^\alpha(1 - \mathrm{i}\varepsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b).$$

Since $|\frac{\theta}{n_1(\rho)}\mu(B(x,r))| \leq \frac{\theta}{n_1(\rho)}|\mu|(\mathbb{R}^d)$, this equivalence relation is uniform both in x and r and can be integrated. This yields

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} \lambda(\rho)\,\Psi_G\big(n_1(\rho)^{-1}\theta\mu(B(x,r))\big)\mathrm{d}x\,F_\rho(\mathrm{d}r) \sim -\sigma^\alpha\rho^{-\beta}|\theta|^\alpha\int_{\mathbb{R}^d\times\mathbb{R}^+}|\mu(B(x,r))|^\alpha(1 - \mathrm{i}\varepsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b)\mathrm{d}x\,F_\rho(\mathrm{d}r). \tag{23}$$

Finally, Lemma 3.2 applies with

$$g(r) = \int_{\mathbb{R}^d}|\mu(B(x,r))|^\alpha(1 - \mathrm{i}\varepsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b)\mathrm{d}x,$$

note that (7) implies that g satisfies condition (14). Consequently,

$$\int_{\mathbb{R}^d\times\mathbb{R}^+}|\mu(B(x,r))|^\alpha(1 - \mathrm{i}\varepsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b)\mathrm{d}x\,F_\rho(\mathrm{d}r) \sim C_\beta\rho^\beta\int_{\mathbb{R}^d\times\mathbb{R}^+}|\mu(B(x,r))|^\alpha(1 - \mathrm{i}\varepsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b)r^{-\beta-1}\mathrm{d}x\mathrm{d}r. \tag{24}$$

Finally, (23) and (24) together imply (22), and as explained at the beginning of Section 3, this proves Theorem 2.4. □

3.4. *Proof of condition* (10)

We prove that Condition (10) for the existence of J is satisfied. Note that this condition splits into:

$$\int_{|m\mu(B(x,r))|\leq 1}(m\mu(B(x,r)))^2 r^{-\beta-1}\mathrm{d}x\mathrm{d}r\,G(\mathrm{d}m) < +\infty \tag{25}$$

and

$$\int_{|m\mu(B(x,r))|\geq 1}|m\mu(B(x,r))|r^{-\beta-1}\mathrm{d}x\mathrm{d}r\,G(\mathrm{d}m) < +\infty. \tag{26}$$

We shall use the following Lemma for the truncated moments of a distribution in the normal domain attraction of a stable law:

Lemma 3.4. *Let G be in the normal domain attraction of an α-stable law for $\alpha > 1$. There are $C_1, C_2 \in (0, +\infty)$ such that for all $x \geq 0$:*

$$\int_{|m|\geq x}|m|G(\mathrm{d}m) \leq C_1 x^{1-\alpha} \quad \textit{and} \quad \int_{-x}^{x} m^2 G(\mathrm{d}m) \leq C_2 x^{2-\alpha}.$$

Proof of Lemma 3.4. From [3, XVII.5], we have $\int_{-x}^{x} m^2 G(\mathrm{d}m) \sim Cx^{2-\alpha}$ when $x \to +\infty$ (note that since G is in the *normal* domain of attraction, there is no slowly varying function in this estimate). But since moreover for $x \in [0, 1]$

$$\int_{-x}^{x} m^2 G(\mathrm{d}m) = \int_0^{x^2} G(m : u \le m^2 \le x^2)\mathrm{d}u \le x^2 \le x^{2-\alpha}$$

and the second part is proved.

Next, since $\lim_{x\to 0} \int_{|m|>x} |m|G(\mathrm{d}m) = \int_{\mathbb{R}} |m|G(\mathrm{d}m) < +\infty$ while $x^{1-\alpha} \to +\infty$, $x \to 0$, the first part comes from [3, Eq. (5.21)]:

$$\int_{|m|>x} |m|G(\mathrm{d}m) \sim \frac{2-\alpha}{\alpha-1}\frac{1}{x}\int_{-x}^{x} m^2 G(\mathrm{d}m) \sim \frac{2-\alpha}{\alpha-1}x^{1-\alpha}, \quad x \to +\infty.$$

Now, we prove (25) and (26). First for (25), we have:

$$\begin{aligned}
&\int_{|m\mu(B(x,r))|\le 1} (m\mu(B(x,r)))^2 r^{-\beta-1}\mathrm{d}x\mathrm{d}r G(\mathrm{d}m) \\
&\le \int_{\mathbb{R}^d\times\mathbb{R}^+} \left(\int_{-1/|\mu(B(x,r))|}^{+1/|\mu(B(x,r))|} m^2 G(\mathrm{d}m)\right) \mu(B(x,r))^2 r^{-\beta-1}\mathrm{d}x\mathrm{d}r \\
&\le C_2 \int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))|^{\alpha-2}\mu(B(x,r))^2 r^{-\beta-1}\mathrm{d}x\mathrm{d}r \\
&\le C_2 \int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))|^{\alpha} r^{-\beta-1}\mathrm{d}x\mathrm{d}r
\end{aligned}$$

which is finite when $\mu \in \mathcal{M}_{\alpha,\beta}$ (see Proposition 2.2-(i)). Next for (26), we have:

$$\begin{aligned}
&\int_{|m\mu(B(x,r))|\ge 1} |m\mu(B(x,r))| r^{-\beta-1}\mathrm{d}x\mathrm{d}r G(\mathrm{d}m) \\
&\le \int_{\mathbb{R}^d\times\mathbb{R}^+} \left(\int_{|m|>1/|\mu(B(x,r))|} |m|G(\mathrm{d}m)\right) |\mu(B(x,r))| r^{-\beta-1}\mathrm{d}x\mathrm{d}r \\
&\le C_1 \int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))|^{\alpha-1}|\mu(B(x,r))| r^{-\beta-1}\mathrm{d}x\mathrm{d}r \\
&\le C_1 \int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))|^{\alpha} r^{-\beta-1}\mathrm{d}x\mathrm{d}r
\end{aligned}$$

which, again, is finite when $\mu \in \mathcal{M}_{\alpha,\beta}$. □

3.5. *Proof of Theorem 2.11*

As in the proof of Theorem 2.4, it is enough to consider convergence of one-dimensional marginals. The characteristic function of the Poisson integral $M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]$ is given by (13) and that of the generalized random field $J(\mu)$ is given by

$$\begin{aligned}
\varphi_{J(\mu_a)}(\theta) &= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} \Psi(\theta m\mu(B(a^{-1}x, a^{-1}r)))C_\beta r^{-1-\beta}\mathrm{d}r\mathrm{d}x G(\mathrm{d}m)\right) \\
&= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G(\theta\mu(B(x,r)))C_\beta a^{d-\beta} r^{-1-\beta}\mathrm{d}r\mathrm{d}x\right).
\end{aligned}$$

From Lemma 3.1, $|\Psi_G(\theta\mu(B(x,r)))| \leq C|\theta|^\alpha|\mu(B(x,r))|^\alpha$ for some $C > 0$, so that condition (14) for $g(r) = \int_{\mathbb{R}^d} \Psi_G(\mu(B(x,r)))\mathrm{d}x$ is given again by (7) when $\mu \in \mathcal{M}_{\alpha,\beta}$. Thus, Lemma 3.2 applies and together with $\lim_{\rho\to 0^{-\epsilon}} \lambda(\rho)\rho^\beta = a^{d-\beta}$ entail

$$\lim_{\rho\to 0^{-\epsilon}} \int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G\left(\theta\mu(B(x,r))\right) \mathrm{d}x\lambda(\rho)F_\rho(\mathrm{d}r)$$
$$= C_\beta a^{d-\beta} \int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G\left(\theta\mu(B(x,r))\right) r^{-\beta-1}\mathrm{d}r\mathrm{d}x.$$

Since one-dimensional convergence is enough, this achieves the proof of Theorem 2.11. □

3.6. *Proof of Proposition* 2.13

We consider the subsequence $a_m = m^{1/(d-\beta)}$. From the aggregate-similarity of the field J (see (11) in Proposition 2.12), we have:

$$\frac{1}{a_m^{(d-\beta)/\alpha}} J(\mu_{a_m}) \overset{f.d.d.}{=} \frac{1}{m^{1/\alpha}} \sum_{i=1}^m J^i(\mu)$$

for independent copies J^i, $1 \leq i \leq m$, of J. But

$$\varphi_{m^{-1/\alpha}\sum_{i=1}^m J^i(\mu)}(\theta) = \left(\varphi_{J(\mu)}(m^{-1/\alpha}\theta)\right)^m$$
$$= \exp\left(m\int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G(m^{-1/\alpha}\theta\mu(B(x,r)))C_\beta r^{-1-\beta}\mathrm{d}r\mathrm{d}x\right),$$

and from Lemma 3.1,

$$\Psi_G(m^{-1/\alpha}\theta\mu(B(x,r))) \sim \sigma^\alpha|\theta|^\alpha|\mu(B(x,r))|^\alpha(1 - \mathrm{i}\epsilon(\theta\mu(B(x,r)))\tan(\pi\alpha/2)b).$$

The relation above is uniform both in x and r and it is thus integrable with respect to $\mathrm{d}r\mathrm{d}x$. This yields

$$\lim_{m\to+\infty} \varphi_{m^{-1/\alpha}\sum_{i=1}^m J^i(\mu)}(\theta) = \exp\left(C_\beta\sigma^\alpha|\theta|^\alpha \int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))|^\alpha\right.$$
$$\left.\times\left(1 - \mathrm{i}\epsilon(\theta\mu(B(x,r))\tan\left(\frac{\pi\alpha}{2}\right)b\right)r^{-1-\beta}\mathrm{d}r\mathrm{d}x\right).$$

A standard argument completes the proof of convergence in distribution along an arbitrary sequences. □

3.7. *Proof of Theorem* 2.16

We follow the argument in the proof of Theorem 2 in [7]. Recall that here $d < \beta < \alpha d$ so that $\epsilon = -1$ and the limits are taken when $\rho \to 0$. Again, by linearity, using the Cramér–Wold device, it is enough to deal with one-dimensional marginals. From (13) with a change of variable, the characteristic function rewrites

$$\varphi_{n_2(\rho)^{-1}(M_\rho(\mu)-\mathbb{E}[M_\rho(\mu)])}(\theta)$$
$$= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G\left(\theta n_2(\rho)^{-1}\mu(B(x, n_2(\rho)^{1/d}r))\right)\lambda(\rho)\mathrm{d}x F_{\rho n_2(\rho)^{-1/d}}(\mathrm{d}r)\right).$$

Let $\mu(dz) = \phi(z)dz$ with $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$, then, from Lemma 4 in [7], as $n_2(\rho) \to 0$,

$$n_2(\rho)^{-1}\mu(B(x, n_2(\rho)^{1/d}r)) \to \phi(x)c_d r^d$$

dx almost everywhere and

$$x \mapsto \phi^*(x) = \sup_{v>0}\left(c_d^{-1}v^{-d}|\mu|(B(x,v))\right) \in L^\alpha(\mathbb{R}^d). \tag{27}$$

As a consequence,

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G\left(\theta n_2(\rho)^{-1}\mu(B(x, n_2(\rho)^{1/d}r))\right)\lambda(\rho)\mathrm{d}x\,F_{\rho n_2(\rho)^{-1/d}}(\mathrm{d}r)$$
$$\sim C_\beta\lambda(\rho)\rho^\beta n_2(\rho)^{-\beta/d}\int_{\mathbb{R}^d\times\mathbb{R}^+}\Psi_G\left(\theta\phi(x)c_d r^d\right)r^{-\beta-1}\mathrm{d}r\mathrm{d}x. \tag{28}$$

To see this, apply Lemma 3.2 to

$$g(r) = \int_{\mathbb{R}^d}\Psi_G\left(\theta\phi(x)c_d r^d\right)\mathrm{d}x$$

and to

$$g_\rho(r) = \int_{\mathbb{R}^d}\Psi_G\left(\theta n_2(\rho)^{-1}\mu(B(x, n_2(\rho)^{1/d}r))\right)\mathrm{d}x.$$

Since $|\Psi_G(u)| \le C(|u| \wedge |u|^\alpha)$, we have

$$|g(r)| \le C\min\left(c_d|\theta|\|\phi\|_{L^1}r^d,\ c_d^\alpha|\theta|^\alpha\|\phi\|_{L^\alpha}^\alpha r^{\alpha d}\right)$$

so that condition (14) is satisfied with $p = d$ and $q = \alpha d$. Furthermore, since Ψ_G is a K-Lipschitzian function for some finite K, we get

$$|g(r) - g_\rho(r)| \le Kc_d r^d|\theta|\int_{\mathbb{R}^d}\left|c_d^{-1}r^{-d}n_2(\rho)^{-1}\mu(B(x, n_2(\rho)^{1/d}r)) - \phi(x)\right|\mathrm{d}x.$$

The integrand $\left|c_d^{-1}r^{-d}n_2(\rho)^{-1}\mu(B(x, n_2(\rho)^{1/d}r)) - \phi(x)\right|$ converges to zero dx almost everywhere. Since its L^α-norm is bounded by $\|\phi^*\|_{L^\alpha} + \|\phi\|_{L^\alpha}$, it is uniformly integrable and as a consequence,

$$\lim_{\rho\to 0}\int_{\mathbb{R}^d}\left|c_d^{-1}r^{-d}n_2(\rho)^{-1}\mu(B(x, n_2(\rho)^{1/d}r)) - \phi(x)\right|\mathrm{d}x = 0.$$

On the other hand, since for some $C > 0$, $|\Psi_G(v)| \le C|v|^\alpha$, we obtain

$$|g(r) - g_\rho(r)| \le C(\|\phi^*\|_{L^\alpha} + \|\phi\|_{L^\alpha})r^{\alpha d}.$$

Hence, g_ρ satisfy condition (15) with $p = d$ and $q = \alpha d$. This proves (28).

From the definition of $n_2(\rho)$, $\lambda(\rho)\rho^\beta n_2(\rho)^{-\beta/d} = 1$. Furthermore, by splitting the integration over $\mathbb{R}^d$ into $\{x \in \mathbb{R}^d : \theta\phi(x) \ge 0\}$ and $\{x \in \mathbb{R}^d : \theta\phi(x) < 0\}$ and performing a change of variable, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^+}\Psi_G\left(\theta\phi(x)c_d r^d\right)r^{-\beta-1}\mathrm{d}r\mathrm{d}x = D\int_{\mathbb{R}^d}(\theta\phi(x))_+^\gamma\mathrm{d}x + \bar{D}\int_{\mathbb{R}^d}(\theta\phi(x))_-^\gamma\mathrm{d}x,$$

where $\bar{D}$ is the complex conjugate of $D = d^{-1}c_d^\gamma \int_{\mathbb{R}^+} \Psi_G(r) r^{-\gamma-1} \mathrm{d}r$. We deduce

$$\varphi_{n_2(\rho)^{-1}(M_\rho(\mu)-\mathbb{E}[M_\rho(\mu)])}(\theta) = \exp\left(-\sigma_\phi^\gamma |\theta|^\gamma \left(1 + \mathrm{i} b_\phi \varepsilon(\theta) \tan\left(\frac{\pi\gamma}{2}\right)\right)\right)$$

where

$$\sigma_\phi^\gamma = \sigma_\gamma^\gamma \int_{\mathbb{R}^d} |\phi(x)|^\gamma \mathrm{d}x,$$

and

$$b_\phi = \frac{\int_{\mathbb{R}_+} r^{-1-\gamma}(r - \sin(r))\mathrm{d}r}{\tan(\pi\gamma/2)\int_{\mathbb{R}_+} r^{-1-\gamma}(1-\cos(r))\mathrm{d}r} \frac{\int_{\mathbb{R}} \varepsilon(m)|m|^\gamma G(\mathrm{d}m)}{\int_{\mathbb{R}} |m|^\gamma G(\mathrm{d}m)} \frac{\int_{\mathbb{R}^d} \varepsilon(\phi(x))|\phi(x)|^\gamma \mathrm{d}x}{\int_{\mathbb{R}^d} |\phi(x)|^\gamma \mathrm{d}x}. \tag{29}$$

But since for $\gamma \in (1, 2)$,

$$\int_0^{+\infty} \frac{\mathrm{e}^{\mathrm{i}xu} - 1 - \mathrm{i}xu}{x^{1+\gamma}} \mathrm{d}x = |u|^\gamma \frac{\Gamma(2-\gamma)}{(1-\gamma)(2-\gamma)} (\cos(\pi\gamma/2) - \mathrm{i}\varepsilon(u)\sin(\pi\gamma/2))$$

see Lemma 2 in [3, XVII.4] (with $p = 1, q = 0$ therein), the first ratio on the right-hand side (29) is -1 and we have $b_\phi = b_\gamma \frac{\int_{\mathbb{R}^d} \varepsilon(\phi(x))|\phi(x)|^\gamma \mathrm{d}x}{\int_{\mathbb{R}^d} |\phi(x)|^\gamma \mathrm{d}x}$ where b_γ is given in (12). This achieves the proof of Theorem 2.16. □

3.8. *Proof of* Theorem 2.19

The argument uses the same tools as in the proof of Theorem 2.16 and we only give here the main lines. From (13) and a change of variable, the characteristic function rewrites

$$\varphi_{n_3(\rho)^{-1}(M_\rho(\mu)-\mathbb{E}[M_\rho(\mu)])}(\theta) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G\big(\theta n_3(\rho)^{-1}\mu(B(x,\rho r))\big)\lambda(\rho)\mathrm{d}x F(\mathrm{d}r)\right).$$

Let $\mu(\mathrm{d}z) = \phi(z)\mathrm{d}z$ with $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$. Since as $\rho \to 0$

$$\theta n_3(\rho)^{-1}\mu(B(x,\rho r)) \sim \lambda(\rho)^{-1/\alpha} c_{\mathrm{d}} r^d \phi(x)$$

and $\lambda(\rho) \to +\infty$

$$\begin{aligned}&\lim_{\rho\to 0} \lambda(\rho)\Psi_G\big(\theta n_3(\rho)^{-1}\mu(B(x,\rho r))\big)\\ &\quad = -\sigma^\alpha c_d^\alpha |\theta|^\alpha r^{\alpha d}|\phi(x)|^\alpha \left(1 - \varepsilon(\theta\phi(x))\tan\left(\frac{\pi\alpha}{2}\right) b\right)\end{aligned}$$

$\mathrm{d}x$ almost everywhere, and this latter function is integrable with respect to $\mathrm{d}x F(\mathrm{d}r)$ since $\phi \in L^\alpha(\mathbb{R}^d)$ and $\int_{\mathbb{R}^+} r^{\alpha d} F(\mathrm{d}r) < +\infty$. Furthermore, with ϕ^* given in (27), we derive the following bound:

$$\begin{aligned}\left|\lambda(\rho)\Psi_G\big(\theta n_3(\rho)^{-1}\mu(B(x,\rho r))\big)\right| &\le \lambda(\rho) C n_3(\rho)^{-\alpha}|\mu(B(x,\rho r))|^\alpha\\ &\le C r^{\alpha d}|\phi^*(x)|^\alpha.\end{aligned}$$

This upper bound is independent of ρ and integrable with respect to $\mathrm{d}x\,F(\mathrm{d}r)$ since $\phi^* \in L^\alpha(\mathbb{R}^d)$. The dominated convergence theorem yields:

$$\lim_{\rho\to 0}\int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G\big(\theta n_3(\rho)^{-1}\mu(B(x,\rho r))\big)\lambda(\rho)\mathrm{d}x\,F(\mathrm{d}r)$$
$$= -\sigma^\alpha c_d^\alpha|\theta|^\alpha\int_{\mathbb{R}^+} r^{\alpha d}F(\mathrm{d}r)\int_{\mathbb{R}^d}|\phi(x)|^\alpha(1-\varepsilon(\theta\phi(x))\tan(\pi\alpha/2)b)\mathrm{d}x.$$

This proves Theorem 2.19. □

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