Data Structures with Dynamical Random Transitions

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ABSTRACT: We present a (*non-standard*) probabilistic analysis of dynamic data structures whose sizes are considered dynamic random walks. The basic operations (insertion, deletion, positive and negative queries, batched insertion, lazy deletion, etc.) are time-dependent random variables. This model is a (small) step toward the analysis of these structures when the distribution of the set of histories is *not uniform*. As an illustration, we focus on list structures (linear lists, priority queues, and dictionaries) but the technique is applicable as well to more advanced data structures. © 2005 Wiley Periodicals, Inc. Random Struct. Alg., 28, 403–426, 2006

Keywords: dynamic random walks; data structures; dynamical systems; large deviation principle; random walk in random environment

1. INTRODUCTION

The integrated time and space costs of sequences of operations on list structures have been estimated by Flajolet, Françon, and Vuillemin [7] by combinatorial methods. Louchard [16] and Maier [19] presented two different probabilistic analyses of these dynamic data structures, which led to the same conclusion: the integrated space and time costs of a sequence

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of n supported operations converge, as n goes to infinity, to Gaussian random variables. All the above-mentioned results have been proved under a set of assumptions that constitute the so-called Markovian model and assuming uniform distribution on the set of histories. Following the conclusions of Knuth [14] about deletions that preserve randomness, Louchard, Randrianarimanana, and Schott [18] have shown how to analyze dynamic data structures in the more realistic model proposed by Knuth. The maxima properties (value and position) of these data structures have been analyzed by Louchard, Kenyon, and Schott [17]. As discussed by Maier [19], the model of equiprobable histories is unrealistic and necessitates rejection. The main purpose of this paper is to derive the asymptotic properties of data structure sizes considered dynamic random walks introduced and studied by the second author [10–12].

In Section 2 we give some preliminaries on dynamic data structures. In Section 3 we define the probabilistic model under consideration that is the dynamic random walk in any dimension; then we recall some of their properties and prove a functional large deviation principle. In Sections 4, 5, and 6, the asymptotic behavior of the dynamic data structures size is studied as well as the storage cost function one. Section 7 is devoted to the example of linear lists when the transformation is a rotation.

2. PRELIMINARIES

A data type is a specification of the basic operations allowed together with its set of possible restrictions. The following data types are commonly used.

Stack. Keys are accessed by position; operations are insertion I and deletion D but are restricted to operate on the key positioned first in the structure (the "top" of the stack).

Linear List. Keys are accessed by position; operations are insertion I and deletion D without access restrictions (linear lists make it possible to maintain dynamically changing arrays).

Dictionary. Keys belonging to a totally ordered set are accessed by value; all four operations I, D, Q^+, Q^- are allowed without any restriction. Q^+ represents a positive (successful) query (or search). Q^- stands for a negative (unsuccessful) query (or search).

Priority Queue. Keys belonging to a totally ordered set are accessed by value; the basic operations are I and D; deletion D is performed only on the key of minimal value (of "highest priority").

Symbol Table. This type is a particular case of dictionary where deletion always operates on the key last inserted in the structure; only positive queries are performed.

A data organization is a machine implementation of a data type. It consists of a data structure that specifies the way objects are internally represented in the machine, together with a collection of algorithms implementing the operations of the data type. Stacks are almost always implemented by arrays or linked lists.

Linear lists are often implemented by linked lists and arrays.

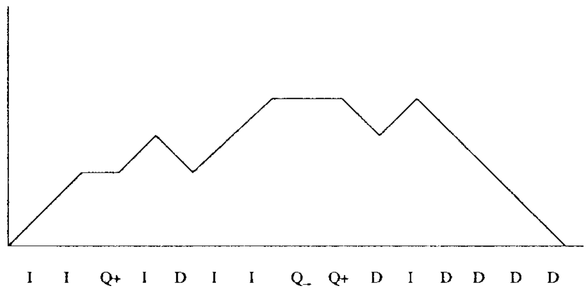

Fig. 1. A schema.

Dictionaries are usually implemented by sorted or unsorted lists; binary search trees have a faster execution time and several balancing schemes have been proposed: AVL, 2-3, and red–black trees. Other alternatives are h-tables and digital trees.

Priority queues can be represented by any of the search trees used for dictionaries; more interesting are heaps, P-tournaments, leftist tournaments, binomial tournaments, binary tournaments, and pagodas. One can also use sorted lists and any of the balanced tree structures.

Symbol tables are special cases of dictionaries; all the known implementations of dictionaries are applicable here.

Definition 2.1. *A schema (or path) is a word*

$$\Omega = O_1 O_2 \cdots O_n \in \{I, D, Q^+, Q^-\}^*$$

such that for all j, $1 \leq j \leq n$*:*

$$|O_1 O_2 \cdots O_j|_I \geq |O_1 O_2 \cdots O_j|_D.$$

$\{I, D, Q^+, Q^-\}^*$ *is the free monoid generated by the alphabet*

$$\{I, D, Q^+, Q^-\}.$$

$|w|$ *is the length of the word w.*

A schema is to be interpreted as a sequence of n requests (the keys operated on not being represented).

Example. Figure 1 shows a schema.

Definition 2.2. *A structure history is a sequence of the form*

$$h = O_1(r_1) O_2(r_2) \cdots O_n(r_n),$$

where $\Omega = O_1 O_2 \cdots O_n$ *is a schema, and the* r_j *are integers satisfying:* $0 \leq r_j < pos(\alpha_{j-1}(\Omega))$ *and* $\alpha_j(\Omega) = |O_1 O_2 \cdots O_j|_I - |O_1 O_2 \cdots O_j|_D$ *is the size (level) of the structure at step j, pos is a possibility function defined on each request, and* r_j *is the rank (or position) of the key operated upon at step j.*

We will only consider schemas and histories with initial and final level 0.

TABLE 1. Possibility Functions in the Markovian Model

Data type	$N\text{pos}(I,k)$	$N\text{pos}(D,k)$	$N\text{pos}(Q^+,k)$	$N\text{pos}(Q^-,k)$
Dictionary	$k+1$	k	k	$k+1$
Priority queue	$k+1$	1	0	0
Linear list	$k+1$	k	0	0

2.1. Possibility Functions

Two different models have been considered for defining possibility functions: first is the markovian model [7, 16] in which possibility functions are linear functions of the size k of the data structure when an allowed operation is performed.

Knuth's model is related to his observation [14] that deletions may not preserve randomness and is more realistic than the markovian model. The following simple example may be helpful to understand Knuth's fundamental remark.

Consider the sequence of operations *IIDI* performed, for example, on a linear list that is initially empty. Let $x < y < z$ be the three keys inserted during the sequence III. x, y, and z are deleted with equal probability. Let w be the key inserted by the fourth I. Then all four cases $w < x < y < z$, $x < w < y < z$, $x < y < w < z$, $x < y < z < w$ do occur with equal probability, whatever the key deleted. More generally, let us consider a sequence of operations $O_1 O_2 \cdots O_j$ on a dictionary data type, the initial data structure being empty (any data type listed above may be considered). Assume O_j is the ith I or Q^- of the sequence. Let $x_1 < x_2 < \cdots < x_{i-1}$ be the keys inserted and negatively searched during the sequence $O_1 O_2 \cdots O_{j-1}$, and let w be the ith inserted or negatively searched key. Then all the cases $w < x_1 < x_2 < \cdots < x_{i-1}$,$x_1 < w < x_2 < \cdots < x_{i-1}, \ldots, x_1 < x_2 < \cdots < x_{i-1} < w$ are equally likely, whatever the deleted keys. Put into combinatorial words: after j operations, whose i are I and Q^-'s (thus $j-i$ are D and Q^+'s), the size of the data structure is $k \leq 2i-j$. The keys of the data structure can be considered a subset of k distinct objects of a set of size i any of the C_i^k possible subsets being equally likely. We say that the number of possibilities of the ith I or Q^- (in a sequence of operations) is equal to i in Knuth's model, whatever the size of the data structure when this insertion (or negative query) occurs. We summarize in Tables 1 and 2 the differences between the markovian and Knuth models. We consider only a few data structures.

If instead of deleting items from the data structure as described above we wait until there is a new insertion, we get a new operation called lazy deletion (see [17] and the references therein). Batched insertion waits until there is a new deletion. The probabilistic model described below applies also for these operations but we restrict our analysis to dynamic data structures subject to the classical operations: $I, D, Q^+,\ Q^-$.

TABLE 2. Possibility Functions in Knuth's Model

Data type	$N\text{pos}(i\text{th } I)$	$N\text{pos}(D,k)$	$N\text{pos}(Q^+,k)$	$N\text{pos}(i\text{th } Q^-)$
Dictionary	i	k	k	i
Priority queue	i	1	0	0
Linear list	i	k	0	0

3. THE PROBABILISTIC MODEL

3.1. Definition

Let $S = (E, \mathcal{A}, \mu, T)$ be a dynamical system where $(E, \mathcal{A}, \mu)$ is a probability space and T is a measure-preserving transformation defined on E. Let $d \geq 1$ and $(e_j)_{1\leq j\leq d}$ be the unit coordinate vectors of $\mathbb{Z}^d$. Let $f_1, \ldots, f_d$ be measurable functions defined on E with values in $[0, \frac{1}{d}]$. For each $x \in E$, we denote by $\mathbb{P}_x$ the distribution of the time-inhomogeneous random walk,

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i \qquad \text{for } n \geq 1,$$

with step distribution

$$\mathbb{P}_x(X_i = z) = \begin{cases} f_j(T^i x) & \text{if } z = e_j \\ \frac{1}{d} - f_j(T^i x) & \text{if } z = -e_j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It is worth remarking that if the functions f_j are not all constant, $(S_n)_{n\in\mathbb{N}}$ is a non-homogeneous Markov chain. This Markov chain can be classified in the large class of random walks evolving in a random environment. In most of the papers (see, for instance, [9], [3]), the environment field takes place in space but it can also take place in space and time (see [2]). Following the formalism used in the study of these random walks, when x is fixed, the measure $\mathbb{P}_x$ is called *quenched* and the measure averaged on values of x defined as $\mathbb{P}(.) = \int_E \mathbb{P}_x(.)\, d\mu(x)$ is called *annealed*. The dynamic random walks were introduced and studied by the second author in [10–12]. Let us recall the results already obtained in the quenched setting. Denote by $A = (a_{ij})_{1\leq i,j\leq d}$ the matrix with coefficients

$$a_{jj} = \frac{1}{d^2} \int_E \left(d + 1 - 4d^2 f_j^2\right) d\mu$$
$$a_{ij} = a_{ji} = \frac{1}{d^2} \int_E (1 - 4d^2 f_i f_j)\, d\mu.$$

Let $\mathcal{C}_1(S)$ be the class of functions $f \in L^1(\mu)$ satisfying the condition for μ-almost every point $x \in E$,

$$\left| \sum_{k=1}^{n} \left(f(T^k x) - \int_E f\, d\mu \right) \right| = o\left(\frac{\sqrt{n}}{\log n} \right).$$

Choose $f_j \in \mathcal{C}_1(S), j = 1, \ldots, d$ such that for every $j, l \in \{1, \ldots, d\}, f_j f_l \in \mathcal{C}_1(S)$, and $\int_E f_j\, d\mu = \frac{1}{2d}$. Then, for μ-almost every point $x \in E$, S_n satisfies a local limit theorem, namely

$$\mathbb{P}_x(S_{2n} = 0) \sim \frac{2}{\sqrt{\det A}(4\pi n)^{\frac{d}{2}}} \text{ as } n \to \infty$$

(see [12]).

Let $\mathcal{C}_2(S)$ denote the class of functions $f \in L^1(\mu)$ satisfying the following condition:

$$\sup_{x \in E} \left| \sum_{i=1}^{n} \left(f(T^i x) - \int_E f \, d\mu \right) \right| = o(\sqrt{n}).$$

Assume that for every $j, l \in \{1, \ldots, d\}$, $f_j \in \mathcal{C}_2(S)$, $f_j f_l \in \mathcal{C}_2(S)$, and $\int_E f_j \, d\mu = \frac{1}{2d}$, then, for every $x \in E$, the sequence of processes $(\frac{1}{\sqrt{n}} S_{[nt]})_{t \geq 0}$ weakly converges in the Skorohod space $\mathcal{D} = \mathcal{D}([0, \infty[)$ (see [1]) to the d-dimensional Brownian motion $B_t = (B_t^{(1)}, \ldots, B_t^{(d)})$ with zero mean and covariance matrix At (see [13]). This result was used to study some problems related to resource sharing, namely distributed algorithms as the well-known colliding stacks problem or the Banker's algorithm in the case when the requests are time dependent (see [13] for further details). Let us also remark that the dynamic $\mathbb{Z}^d$-random walks quite differ from the standard $\mathbb{Z}^d$-random walks on nearest neighbors since the matrix A is not necessarily diagonal and some dimensional correlations appear in the limit process $(B_t)_{t \geq 0}$. A strong law of large numbers for the dynamic random walks can be obtained for μ-almost every point $x \in E$ from Kolmogorov's theorem assuming that the functions $f_1, \ldots, f_d$ are measurable. The limit vector is then given by $(2\mu(f_j|\mathcal{I}) - 1)_{1 \leq j \leq d}$ where $\mathcal{I}$ is the invariant σ-field associated to the transformation T. So, $(S_n/n)_{n \geq 1}$ is a good candidate for the large deviation principle (LDP). In the next section, we derive a functional LDP for the dynamic $\mathbb{Z}^d$-random walks.

3.2. Large Deviation Principles

Let Γ be a Polish space endowed with the Borel σ-field $\mathcal{B}(\Gamma)$. A good *rate* function is a lower semi-continuous function $\Lambda^* : \Gamma \to [0, \infty]$ with compact level sets $\{x; \Lambda^*(x) \leq \alpha\}, \alpha \in [0, \infty[$. Let $v = (v_n)_n \uparrow \infty$ be an increasing sequence of positive reals. A sequence of random variables $(Y_n)_n$ with values in Γ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy *a large deviation principle* with speed $v = (v_n)_n$ and the good rate function Λ^* if for every Borel set $B \in \mathcal{B}(\Gamma)$,

$$-\inf_{x \in B^o} \Lambda^*(x) \leq \liminf_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \leq \limsup_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \leq -\inf_{x \in \bar{B}} \Lambda^*(x).$$

In the following, $f_1, \ldots, f_d$ are functions defined on E with values in $[0, \frac{1}{d}]$. We define the family $(l_\lambda)_{\lambda \in \mathbb{R}^d}$ of functions defined on E with values in $\mathbb{R}$ by

$$l_\lambda := \log \left(\sum_{j=1}^{d} \left(e^{\lambda_j} f_j + \left(\frac{1}{d} - f_j \right) e^{-\lambda_j} \right) \right).$$

For every λ, the function l_λ is bounded by $\log(\frac{2}{d}(\sum_{j=1}^{d} \cosh \lambda_j))$. It is measurable (resp. continuous) as soon as the functions (f_j) are measurable (resp. continuous).

Theorem 3.1. *For μ-a.e. $x \in E$, the distributions of S_n/n under $\mathbb{P}_x$ satisfy the LDP with speed n and the good rate function*

$$\Lambda^*(y) = \sup_{\lambda \in \mathbb{R}^d} \{< \lambda, y > - \Lambda(\lambda)\},$$

where

$$\Lambda(\lambda) = \mu(l_\lambda | \mathcal{I}),$$

$\mathcal{I}$ being the σ-field generated by the fixed points of the transformation T.

Let us assume E to be a compact metric space, $\mathcal{A}$ the associated Borel σ-field, and T a continuous transformation of E. If there exists an unique invariant measure μ, i.e., $(E, \mathcal{A}, \mu, T)$ is uniquely ergodic and if $f_1, \ldots, f_d$ are continuous, then the assertion of Theorem 3.1 holds for every $x \in E$. In that case,

$$\Lambda(\lambda) = \mu(l_\lambda).$$

The rate function is deterministic. Under these stronger hypotheses on the dynamical system, we can extend Theorem 3.1 as follows.

Let us define for every $n \geq 1$,

$$S_n^*(t) := \frac{S_{[nt]}}{n}, \quad t \in [0, 1].$$

The linear interpolation of $S_n^*(t)$, $t \in [0, 1]$, is then defined by

$$\bar{S}_n(t) := S_n^*(t) + \left(t - \frac{[nt]}{n}\right) X_{[nt]+1}.$$

Let $\mathbb{P}_{x,n}$ and $\bar{\mathbb{P}}_{x,n}$ be the distribution of $S_n^*(.)$ and $\bar{S}_n(.)$ in $L_\infty([0, 1])$ equipped with the supremum norm. Throughout, $\mathcal{C}([0, 1])$ denotes the space of continuous functions on $[0, 1]$ and $\mathcal{AC}([0, 1])$ denotes the space of absolutely continuous functions on $[0, 1]$.

Theorem 3.2. *Let $(E, \mathcal{A}, \mu, T)$ be an uniquely ergodic dynamical system. If $f_1, \ldots, f_d$ are continuous, then for every $x \in E$, the distributions $(\bar{\mathbb{P}}_{x,n})_{n\geq 1}$ satisfy in $\mathcal{C}([0, 1])$ equipped with the supremum norm the LDP with the good rate function*

$$I(\phi(.)) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi}(t))\, dt, \ \textit{if}\ \phi(.) \in \mathcal{AC}, \phi(0) = 0 \\ +\infty, \ \textit{otherwise.} \end{cases}$$

Remark. Let us mention that an annealed large deviations statement can easily be proved using results of [5] (E is assumed to be compact). Details are omitted since the dynamic operations on the data structures will be modeled from the quenched probability measure.

3.3. Proof of Theorem 3.1

By independence of (X_i), we have, for every $\lambda \in \mathbb{R}^d$ and every $x \in E$,

$$\frac{1}{n} \log \mathbb{E}_x(e^{<\lambda, S_n>}) = \frac{1}{n} \sum_{i=1}^{n} l_\lambda(T^i x).$$

Therefore, Birkoff's theorem implies that for μ-a.e. $x \in E$ and every $\lambda \in \mathbb{R}^d$,

$$\frac{1}{n} \log \mathbb{E}_x(e^{<\lambda,S_n>}) \to \mu(l_\lambda | \mathcal{I}) = \Lambda(\lambda).$$

Λ being finite and differentiable on $\mathbb{R}^d$, by Gärtner–Ellis theorem (see [4]), the theorem follows.

3.4. Proof of Theorem 3.2

The proof follows the same lines of argument as in [4, section 5.1]. It is a Mogulskii-like theorem. The only difference lies in the proof of the LDP for the finite dimensional marginals, which we claim now.

Proposition 3.1. *Let $\mathcal{P}$ be the set of all ordered finite subsets of the interval* $]0,1]$ *that is the set of k-tuples $t^k = \{t_0 = 0 < t_1 < t_2 < \cdots < t_k \leq 1\}$ with $k \geq 1$. Let $f : [0,1] \to \mathbb{R}^d$; for every $k \geq 1$, for every k-tuple t^k, we define*

$$p_{t^k}(f) = (f(t_1), \ldots, f(t_k)) \in (\mathbb{R}^d)^k.$$

Then, the laws $\bar{\mathbb{P}}_{x,n} \circ p_{t^k}^{-1}$ satisfy in $(\mathbb{R}^d)^k$ the LDP with the good rate function

$$\Lambda_k^*(y) = \sum_{l=1}^{k} (t_l - t_{l-1}) \Lambda^* \left(\frac{y_l - y_{l-1}}{t_l - t_{l-1}} \right).$$

Proof. The difference with the classical proof (see [4], Lemma 5.1.8., p. 178) is that the increments of the dynamic random walk are not stationary. But we can remark that for every $l \geq 1$, an increment of the dynamic random walk $S_{[nt_l]} - S_{[nt_{l-1}]}$ is a new dynamic random walk associated to the dynamical system $(E, \mathcal{A}, \mu, T)$, to the functions $f_1, \ldots, f_d$, and to the point $T^{[nt_l - 1]}x$. Then the hypothesis of unique ergodicity on the dynamical system and Theorem 3.1 permits us to deduce the LDP for the laws $\mathbb{P}_{x,n} \circ p_{t^k}^{-1}$. ∎

3.5. A Riemannian Dynamic Random Walk

Let $f_1, \ldots, f_d$ be one-periodic functions defined on $\mathbb{R}$ with values in $[0, \frac{1}{d}]$ and $(X_{i,n})_{1 \leq i \leq n}$ be a sequence of independent random vectors with values in $\mathbb{Z}^d$ with distribution

$$\mathbb{P}_x(X_{i,n} = z) = \begin{cases} f_j(x + \frac{i}{n}) & \text{if } z = e_j \\ \frac{1}{d} - f_j(x + \frac{i}{n}) & \text{if } z = -e_j \\ 0 & \text{otherwise} \end{cases}.$$

We write

$$S_0 = 0, \; S_n = \sum_{i=1}^{n} X_{i,n} \qquad \text{for } n \geq 1$$

this n-dynamic random walk. This random walk is more difficult to study due to the presence of n in the transition probabilities, which creates much more temporal inhomogeneity. The proof of the following proposition is straightforward.

Proposition 3.2. *Let f be a one-periodic function that can be expanded into a Fourier series $f(x) = \sum_{h\in\mathbb{Z}} c_h e^{2\pi ihx}$.*

When there exists $\beta > 1$ such that $|c_n| + |c_{-n}| = \mathcal{O}(n^{-\beta})$, then

$$\sup_{x\in[0,1]} \left| \sum_{i=1}^{n} \left(f\left(x + \frac{i}{n}\right) - \int_{[0,1]} f(t)\, dt \right) \right| = \mathcal{O}(n^{1-\beta}).$$

When the functions $f_1, \ldots, f_d$ can be expanded into a Fourier series

$$f_j(x) = \sum_{h\in\mathbb{Z}} c_h^{(j)} e^{2\pi ihx}$$

where the coefficients $(c_h^{(j)})_{h\in\mathbb{Z}}$ satisfy $c_0^{(j)} = \frac{1}{2d}$ and there exists $\beta_j > 1$ such that

$$|c_n^{(j)}| + |c_{-n}^{(j)}| = \mathcal{O}(n^{-\beta_j}),$$

then the proof of Theorem 3.2 can be adapted so as to get a functional LDP for the $n-$dynamic $\mathbb{Z}^d$-random walks.

4. DYNAMIC LINEAR LISTS

From Table 1, the evolution of dynamic linear lists is modeled by the one-dimensional dynamic random walk $(S_k)_{0\le k\le n}$, each path being assigned relative weight

$$\prod_{i=1}^{n} (S_{i-1} + 1)$$

and conditioned to end in 0. We must indeed take into account the number of places where we can delete or insert an item in the list. Let $(S_{ll,n}^w)_{0\le k\le n}$ be the weighted random walk and

$$S_{ll,n}^w(t) := \frac{S_{ll,[nt]}^w}{n},\ t \in [0, 1].$$

We use here the same notation as in Section 3 with $d = 1$. We assume that the system is uniquely ergodic. The function f_1 is denoted by f and we assume

$$\int_E |\log[f(1-f)]|\, d\mu < \infty. \tag{2}$$

Let us recall that $\mathbb{P}_{x,n}$ denotes the distribution of the random variable $(\frac{S_{[nt]}}{n})_{t\in[0,1]}$. For every $n \ge 1$, we define the functional $F_n : \Omega \longrightarrow [-\infty, +\infty)$ by

$$F_n(\phi(.)) = \begin{cases} \int_0^1 \log\,(\phi(t) + 1/n)\, dt, \text{ if } \phi(1) = 0. \\ -\infty, \text{ otherwise,} \end{cases}$$

as well as the functional

$$F(\phi(.)) = \begin{cases} \int_0^1 \log\ \phi(t)\, dt, \ \text{if}\ \phi(1) = 0 \\ -\infty, \ \text{otherwise.} \end{cases}$$

With this notation, let us define a probability measure on $L_\infty([0, 1])$ by

$$\mathbb{Q}_{x,n}^{(ll)}(A) = \frac{\int_A \exp(nF_n(\phi))\, d\mathbb{P}_{x,n}}{\int_\Omega \exp(nF_n(\phi))\, d\mathbb{P}_{x,n}}$$

or, in other words,

$$\mathbb{Q}_{x,n}^{(ll)}(A) = \frac{\mathbb{E}_x\big(\big[\prod_{i=1}^n (S_{i-1}+1)\big] 1_{S_n(.)\in A} \big| S_n = 0\big)}{\mathbb{E}_x\big(\big[\prod_{i=1}^n (S_{i-1}+1)\big] \big| S_n = 0\big)}.$$

Let us stress that this probability forces the path to remain nonnegative.

Lemma 4.1. *The functional $I - F$ has an unique minimizer denoted by ϕ_{ll}. This function is absolutely continuous, concave, and satisfies $\dot{\phi}_{ll}(0) > 0$, $\dot{\phi}_{ll}(1) < 0$ and $\phi_{ll} > 0$ on $(0, 1)$. Furthermore, assume that ϕ_{ll} has a continuous second-order derivative. Let $L_{ll}(x, y) = \Lambda^*(y) - \log x$. Then the minimizer ϕ_{ll} is a solution of the Euler–Lagrange equation*

$$\frac{d}{dt}\frac{\partial L_{ll}}{\partial y}(\phi, \dot{\phi}) - \frac{\partial L_{ll}}{\partial x}(\phi, \dot{\phi}) = 0 \tag{3}$$

with boundary conditions $\phi(0) = \phi(1) = 0$.

The proof of this lemma is in the Appendix.

Theorem 4.1. *For $x \in E$, the sequence $(S_{ll,n}^w)$ converges in $\mathbb{Q}_{x,n}^{(ll)}$-probability to ϕ_{ll} as n goes to infinity and this convergence is exponential: for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all sufficiently large n,*

$$\mathbb{Q}_{x,n}^{(ll)}(\{\phi \in \Omega \mid \|\phi - \phi_{ll}\|_\infty \geq \varepsilon\}) \leq \exp(-n\delta).$$

Proof. The proof is essentially based on an adaptation in our particular case of Varadhan's integral lemma (see Section 4.3 in [4]). It will be deduced from both of the following inequalities. For any closed set $A \subset \Omega$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\int_A \exp(nF_n)\, d\mathbb{P}_{x,n}\right] \leq -\inf_{\phi\in A}(I - F)(\phi) \tag{4}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \left[\int_\Omega \exp(nF_n)\, d\mathbb{P}_{x,n}\right] \geq -\inf_{\phi\in\Omega}(I - F)(\phi) = F(\phi_{ll}) - I(\phi_{ll}), \tag{5}$$

where I is given in Theorem 3.2. ∎

Proof of (4). It is a consequence of the proof of Lemma 4.3.6 of [4]. From the one hand, F_n is bounded above by 2. From the other hand, for fixed n, F_n is upper semicontinuous. The dependence of F_n on n needs a slight adaptation of the proof, which is easy. ∎

Proof of (5). The functions F_n and F are not lower semi-continuous, so we are not able to use Lemma 4.3.4 of [4] to establish the lower bound. Let $B(\epsilon) = \{\phi : \phi > \phi_{ll} \text{ on } [\epsilon, 1-\epsilon]\}$. We have

$$\begin{aligned}
\mathbb{E}_{x,n}(\exp(nF_n)) &\geq \int_{B(\epsilon)} \exp(nF_n(\phi))\, d\mathbb{P}_{x,n}(\phi) \\
&= \int_{B(\epsilon)} \exp\left(n \int_0^1 \log\, (\phi(t) + 1/n)\, dt\right)\, d\mathbb{P}_{x,n}(\phi) \\
&\geq \exp\left(n \int_{\varepsilon}^{1-\varepsilon} \log\, (\phi_{ll}(t) + 1/n)\, dt\right) \\
&\quad \times \int_{B(\epsilon)} \exp\left(n \int_0^{\varepsilon} \log\, (\phi(t) + 1/n)\, dt\right) \exp\left(n \int_{1-\varepsilon}^1 \log\, (\phi(t) + 1/n)\, dt\right) \\
&\quad \times d\mathbb{P}_{x,n}(\phi).
\end{aligned}$$

In the last integral, let us condition on $\phi(\varepsilon)$ and $\phi(1-\varepsilon)$ and apply the Markov property. This yields

$$\begin{aligned}
&\int_{B(\epsilon)} \exp\left(n \int_0^{\varepsilon} \log\, (\phi(t) + 1/n)\, dt\right) \exp\left(n \int_{1-\varepsilon}^1 \log\, (\phi(t) + 1/n)\, dt\right) d\mathbb{P}_{x,n}(\phi) \\
&\quad = \mathbb{E}_{x,n}[\, Y_1\ Y_2\ Y_3\,],
\end{aligned}$$

where

$$\begin{aligned}
Y_1 &= \mathbb{P}_{x,n}[B(\varepsilon) \mid \phi(\varepsilon),\ \phi(1-\varepsilon)], \\
Y_2 &= \mathbb{E}_{x,n}\left[\exp\left(n \int_0^{\varepsilon} \log\, (\phi(t) + 1/n)\, dt\right) \mid \phi(\varepsilon)\right], \\
Y_3 &= \mathbb{E}_{x,n}\left[\exp\left(n \int_{1-\varepsilon}^1 \log\, (\phi(t) + 1/n)\, dt\right) \mid \phi(1-\varepsilon)\right].
\end{aligned}$$

If $\phi(\varepsilon) \leq \phi_{ll}(\varepsilon)$ or $\phi(1-\varepsilon) \leq \phi_{ll}(1-\varepsilon)$, then $Y_1 = 0$.

Let us define

$$K_n^{\varepsilon} = \inf_{y \geq \phi_{ll}(\varepsilon)} \mathbb{E}_{x,n}\left[\exp\left(n \int_0^{\varepsilon} \log(\phi(t) + 1/n)\, dt\right) \mid \phi(\varepsilon) = y\right]$$

and

$$L_n^{\varepsilon} = \inf_{y \geq \phi_{ll}(1-\varepsilon)} \mathbb{E}_{x,n}\left[\exp\left(n \int_{1-\varepsilon}^1 \log(\phi(t) + 1/n)\, dt\right) \mid \phi(1-\varepsilon) = y\right].$$

Then, $Y_1\ Y_2\ Y_3\ \geq Y_1\ K_n^{\varepsilon}\ L_n^{\varepsilon}$ and

$$\mathbb{E}_{x,n}[Y_1\ Y_2\ Y_3] \geq \mathbb{E}_{x,n}[Y_1]\ K_n^{\varepsilon}\ L_n^{\varepsilon} = \mathbb{P}_{x,n}(B(\varepsilon))\ K_n^{\varepsilon} L_n^{\varepsilon}.$$

This yields

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{x,n}(\exp(nF_n)) \geq \int_{\varepsilon}^{1-\varepsilon} \log\ \phi_{ll}(t)\ dt + \liminf_{n\to\infty} \frac{1}{n} \log\ \mathbb{P}_{x,n}(B(\varepsilon))$$
$$+ \liminf_{n\to\infty} \frac{1}{n} \log K_n^{\varepsilon} + \liminf_{n\to\infty} \frac{1}{n} \log L_n^{\varepsilon}.$$

The set $B(\varepsilon)$ is open, so from the lower bound of large deviations (Theorem 3.2), we get

$$\liminf_{n\to\infty} \frac{1}{n} \log\ \mathbb{P}_{x,n}(B(\varepsilon)) \geq -\inf\{I(\phi); \phi \in B(\varepsilon)\}.$$

Now, thanks to the regularity of I,

$$\inf\{I(\phi); \phi \in B(\varepsilon)\} = \inf\{I(\phi); \phi \in \bar{B}(\varepsilon)\} \leq I(\phi_{ll}).$$

Letting $\varepsilon \to 0$, this yields

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{x,n}(\exp(nF_n))$$
$$\geq F(\phi_{ll}) - I(\phi_{ll}) + \limsup_{\varepsilon\to 0}\ \liminf_{n\to\infty} \frac{1}{n} \log K_n^{\varepsilon} + \limsup_{\varepsilon\to 0}\ \liminf_{n\to\infty} \frac{1}{n} \log L_n^{\varepsilon}.$$

Let $y > \phi_{ll}(\varepsilon) \neq 0$. The conditional expectation

$$\mathbb{E}_{x,n}\left[\exp\left(n\int_0^{\varepsilon} \log\ (\phi(t) + 1/n)\, dt\right)\ |\ \phi(\varepsilon) = y\right]$$

is underestimated by the contribution of the path having increments $+1$ between times 0 and $([n\varepsilon] + ny)/2$ and increments -1 between times $([n\varepsilon] + ny)/2$ and $[n\varepsilon]$. For this path, $\phi(t) + 1/n > \phi_{ll}(\varepsilon)t$ on $[0, \varepsilon]$; hence,

$$\mathbb{E}_{x,n}\left[\exp\left(n\int_0^{\varepsilon} \log\ (\phi(t) + 1/n)\ dt\right)\ |\ \phi(\varepsilon) = y\right]$$
$$\geq \exp\left(n\int_0^{\varepsilon} \log\ (\phi_{ll}(\varepsilon)t)\ dt\right) \prod_{i=1}^{([n\varepsilon]+ny)/2} f(T^i x) \prod_{i=([n\varepsilon]+ny)/2+1}^{[n\varepsilon]} (1 - f(T^i x)).$$

Thus,

$$\frac{1}{n} \log K_n^{\varepsilon} \geq \int_0^{\varepsilon} \log\ (\phi_{ll}(\varepsilon)t)\, dt + \frac{1}{n}\sum_{i=1}^{[n\varepsilon]} \log f(T^i x) + \frac{1}{n}\sum_{i=1}^{[n\varepsilon]} \log\ (1 - f(T^i x)).$$

Then, the uniform ergodicity of the dynamical system and hypothesis (2) imply that

$$\liminf_{n\to\infty} \frac{1}{n} \log K_n^{\varepsilon} \geq \int_0^{\varepsilon} \log\ (\phi_{ll}(\varepsilon)t)\, dt + \varepsilon \int_E \log\, [f(1-f)]\, d\mu.$$

Finally, since $\phi_{ll}(0) = 0$ and $\dot{\phi}_{ll}(0) > 0$ (see Lemma 4.1),

$$\limsup_{\varepsilon\to 0}\ \liminf_{n\to\infty} \frac{1}{n} \log \left(K_n^{\varepsilon}\right) = 0.$$

In the same way, we prove that

$$\limsup_{\varepsilon\to 0}\ \liminf_{n\to\infty} \frac{1}{n} \log\left(L_n^{\varepsilon}\right) = 0.$$

This proves inequality (5).

Let us end the proof of Theorem 4.1. Inequalities (4) and (5) imply that $\mathbb{Q}_{x,n}^{(ll)}$ satisfies a large deviation upper bound: for every closed set $A \subset \Omega$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{Q}_{x,n}^{(ll)}(A) \le -\inf_{\phi\in A}\big[I(\phi) - F(\phi) - \inf_{\phi\in\Omega}(I(\phi) - F(\phi))\big].$$

Apply this inequality with $A_\varepsilon = \{\phi \in \Omega \mid \|\phi - \phi_{ll}\|_\infty \ge \varepsilon\}$. The function F is upper semi-continuous. On the set where the good rate function I is finite, the function F is bounded above (because $I(\phi) < +\infty$ implies $|\dot{\phi}| \le 1$). Thus, by applying the result of Exercise 4.3.10 in [4], we deduce that for any closed set $C \subseteq \Omega$, the infimum of $I - F$ on the set C is attained. Since ϕ_{ll} does not belong to A_ε, the infimum over A_ε is positive. So there exists $\delta = \delta(\varepsilon) > 0$ such that for all sufficiently large n,

$$\mathbb{Q}_{x,n}^{(ll)}(A_\varepsilon) \le \exp(-n\delta). \qquad \blacksquare$$

Remark. Routine calculations give, for every λ,

$$\Lambda'(\lambda) = \int_E \frac{fe^{\lambda} - (1-f)e^{-\lambda}}{fe^{\lambda} + (1-f)e^{-\lambda}}\, d\mu,$$

$$\Lambda''(\lambda) = 4\int_E \frac{f(1-f)}{[fe^{\lambda} + (1-f)e^{-\lambda}]^2}\, d\mu.$$

Under assumption (2), we have $\int_E f(1-f)\, d\mu > 0$ so that $\Lambda'' > 0$ and Λ' is a homeomorphism from $\mathbb{R}$ to $(-1, 1)$. Moreover,

$$(\Lambda^*)'' = ((\Lambda')^{-1})'.$$

Equation (3) is then equivalent to

$$\ddot{\phi}(t)((\Lambda')^{-1})'(\dot{\phi}(t)) = -\frac{1}{\phi(t)} \tag{6}$$

with boundary conditions $\phi(0) = \phi(1) = 0$.

When $f \equiv 1/2$, we have $\Lambda(\lambda) = \log\cosh\lambda$; hence, the unique solution of (6) with boundary conditions, $\phi(0) = \phi(1) = 0$, is given by

$$\phi_{ll}(t) = \frac{1}{\pi}\sin(\pi t)$$

(see [16] or [19] for further details). This example is the simplest one. For other particular functions f, a solution of the above functional equation can perhaps be obtained with numerical methods. This question is hard and is actually under consideration. We will give in Section 7 an example of function f where direct calculations lead to a degenerate nonlinear partial differential equation.

Consider the storage cost function

$$C_{ll,n} = n \sum_{i=1}^{n} S^w_{ll,n}\left(\frac{i}{n}\right).$$

The next result is easily derived from Theorem 4.1.

Corollary 4.1. *Under hypothesis (2), for any $x \in E$, the random variables $(\frac{C_{ll,n}}{n^2})_{n\geq 1}$ converge exponentially fast to*

$$m_{ll} = \int_0^1 \phi_{ll}(t)\, dt$$

as n goes to infinity.

Remark. When $f \equiv 1/2$, under the assumption that ϕ_{ll} is $\mathcal{C}^2$, then $\phi_{ll}(t) = \frac{1}{\pi}\sin(\pi t)$ and $m_{ll} = \frac{2}{\pi^2}$.

5. DYNAMIC PRIORITY QUEUES

This section deals more briefly with priority queues driven by the dynamic random walk defined in Section 3. From Table 1, we see that the difference between dynamic priority queues and dynamic linear lists is the weight we assign to each path. These queues are modeled by the one-dimensional dynamic random walk (S_k), each path being assigned relative weight, and conditioned on $S_n = 0$. It comes from the fact that in the priority queue case, the number of insertions is equal to the number of deletions, with the structure beginning and ending empty. We shall denote this weighted random walk by $(S^w_{pq,n})_{0\leq k\leq n}$. The normalized data structure size as a function of time is defined by

$$S^w_{pq,n}(t) := \frac{S^w_{pq,[nt]}}{n}, \qquad t \in [0,1].$$

The distribution of the random variable $(S^w_{pq,n}(t))_{t\in[0,1]}$ with values in Ω is denoted by $\mathbb{Q}^{(pq)}_{x,n}$.

The situation is similar to Section 4, with, instead of L_{ll}, the function

$$L_{pq}(x,y) = \Lambda^*(y) - \frac{1}{2}\log x,$$

so that the associated Euler–Lagrange equation is

$$\ddot{\phi}(t)(\Lambda^*)''(\dot{\phi}(t)) = -\frac{1}{2\phi(t)},$$

whose solution is denoted by ϕ_{pq}. The storage cost is

$$C_{pq,n} = n \sum_{i=1}^{n} S^w_{pq,n}\left(\frac{i}{n}\right).$$

We have again exponential convergence of $S^w_{pq,n}$ to ϕ_{pq} and of $\frac{C_{pq,n}}{n^2}$ to

$$m_{pq} = \int_0^1 \phi_{pq}(t)\,dt.$$

Remark. When $f \equiv 1/2$, under the assumption that ϕ_{pq} is $\mathcal{C}^2$, then $\phi_{pq}(t) = t(1-t)$ and $m_{pq} = \frac{1}{6}$.

6. DYNAMIC DICTIONARIES

6.1. A New Dynamic Random Walk

Because of the operations supported by dictionaries, we need a different model of dynamic random walks.

We keep the notation of the previous section, except that for each $x \in E$ and $i \geq 1$ the law of X_i is

$$\mathbb{P}_x(X_i = z) = \begin{cases} \frac{1}{2} f(T^i x) & \text{if } z = 1 \\ \frac{1}{2} f(T^i x) & \text{if } z = -1 \\ 1 - f(T^i x) & \text{if } z = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the same results as above hold, with

$$\Lambda_d(\lambda) = \int_E \log(1 + (\cosh(\lambda) - 1)f)\,d\mu.$$

6.2. Large Deviation Principles

All results of Section 3.2 (Theorems 3.1 and 3.2) hold with, instead of Λ,

$$\Lambda_d(\lambda) = \mu(\log(1 + (\cosh(\lambda) - 1)f) \mid \mathcal{I}).$$

When the system is uniquely ergodic, we get

$$\Lambda_d(\lambda) = \mu(\log(1 + (\cosh(\lambda) - 1)f)).$$

In the same way, Theorem 4.1 and Corollary 4.1 hold, up to a change of notation. The Euler–Lagrange equation is now

$$\ddot{\phi}(t)(\Lambda_d^*)''(\dot{\phi}(t)) = -\frac{1}{\phi(t)}, \tag{7}$$

where

$$\Lambda_d^*(y) = \sup_\lambda\{\lambda y - \Lambda_d(\lambda)\}.$$

Remark. When $f \equiv 1/2$, we have $\Lambda_d^* = 2\Lambda^*$ (see [16] or [19]) and, thus, under the assumption that ϕ_d is $\mathcal{C}^2$, the solution ϕ_d of Eq. (7) is the same as in the case of priority queues, namely

$$\phi_d(t) = t(1-t), t \in [0,1]$$

and $m_{pq} = m_d = \frac{1}{6}$.

7. AN EXAMPLE: LINEAR LISTS AND ROTATION ON THE TORUS

Let $([0,1], \mathcal{B}([0,1]), \lambda, T_\alpha)$ be the dynamical system where T_α is defined by $x \to x + \alpha \mod 1$, with α a given real and λ is the Lebesgue measure on $[0,1]$. This particular dynamical system is the so-called rotation on the one-dimensional torus. Twofold motivations are related to this example:

1. Explicit calculations are possible in this case,
2. When α is rational, we get a periodic dynamical system that models a periodic behavior of the operations on the data structures.

Irrational rotations are uniquely ergodic, i.e., the ergodic average of a continuous function uniformly converges in $x \in [0,1]$ to the integral of f. The uniform convergence of the ergodic averages even holds for any function with bounded variation (see [15]). Consequently, if we consider a dynamic random walk associated to an irrational rotation on the torus, Theorem 4.1 holds for every function f with bounded variation.

7.1. Choice of a Function and Derivation of the Corresponding Differential Equation

When hypothesis (2) is not satisfied by function f, the methods presented in the paper do not apply. For instance, let us choose the function $f = 1_{[0,\frac{1}{2}]}$, then $\Lambda = 0$ and

$$\Lambda^*(y) = \begin{cases} +\infty & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In that case, no large deviation occurs as the dynamic random walk is deterministic in the sense that

$$S_n = \sum_{i=1}^{n} \left(2 1_{[0,\frac{1}{2}]}\left(T_\alpha^i x\right) - 1\right)$$

for some $x \in [0,1]$ fixed.

The computation of the function Λ^* for a general function f is difficult (see the remark following Theorem 4.1). However, we are able to compute it for the very particular function $f = \frac{3}{4} 1_{[0,\frac{2}{3}]}$. Unfortunately, this function does not satisfy hypothesis (2). We think that hypothesis (2) is only technical and should be dropped out but we don't have any proof of this point yet. In order to illustrate how our method is efficient to determine the asymptotic behavior of the storage cost function associated to a dynamic linear list, we now compute the function ϕ_{ll} for this particular function f, under the assumption that ϕ_{ll} is $\mathcal{C}^2$. Straightforward computations give us

$$\Lambda(\lambda) = \log(\cosh(\lambda)) + \frac{2}{3}\log\left(1 + \frac{\tanh(\lambda)}{2}\right) + \frac{1}{3}\log(1 - \tanh(\lambda)).$$

When $y \in]-1, \frac{1}{3}[$,

$$\Lambda^*(y) = \frac{1}{2}(1+y)\log(1+y) + \frac{(1-3y)}{6}\log(1-3y)$$

and $+\infty$ otherwise.

We consider dynamic linear lists driven by this particular dynamic random walk and determine the path ϕ_{ll} satisfying the following Euler–Lagrange equation:

$$\ddot{\phi}(t)(\Lambda^*)''(\dot{\phi}(t)) = -\frac{1}{\phi(t)}. \tag{8}$$

In this particular case, on the interval $]-1, 1/3[$,

$$(\Lambda^*)''(y) = ((\Lambda')^{-1})'(y) = \frac{1}{2(1+y)} + \frac{3}{2(1-3y)}.$$

After straightforward computations, Eq. (8) becomes

$$2\phi\ddot{\phi} = 3\dot{\phi}^2 + 2\dot{\phi} - 1. \tag{9}$$

7.2. Study of the Differential Equation (9)

Rewriting the equation like a system of two differential equations with $\theta = \dot{\phi}$ gives

$$\begin{cases} \dot{\phi} = \theta \\ \dot{\theta} = \frac{3}{2\phi}(\theta - \frac{1}{3})(\theta + 1) \end{cases} \iff \dot{z} = F(z) \text{ with } z(t) = \begin{bmatrix} \phi(t) \\ \theta(t) \end{bmatrix},$$

which shows that a solution orbit $(\phi(t), \theta(t))$ may cross the $\phi = 0$ line only when $\theta = \frac{1}{3}$ or $\theta = -1$. We note also that the curves $(\phi(t) = \frac{1}{3}(t - t_0) + \phi_0, \theta(t) = \frac{1}{3})$ and $(\phi(t) = -(t - t_0) + \phi_0, \theta(t) = -1)$ are solutions of the equation; moreover, they are the only polynomial solutions.

A first overview of the behavior of the differential equation may be suggested by plotting the (normalized) vector field F on a grid in the phase space; see Fig. 2 (the letters r,l stand for "right" and "left" and the letters u,m and d for "upper," "middle," and "down"; also x and y are used in place of (respectively) ϕ and θ).

The vector field suggests a symmetry of the trajectories (see Fig. 2). Furthermore, we have also a similarity principle between some solution orbits. More precisely, we have

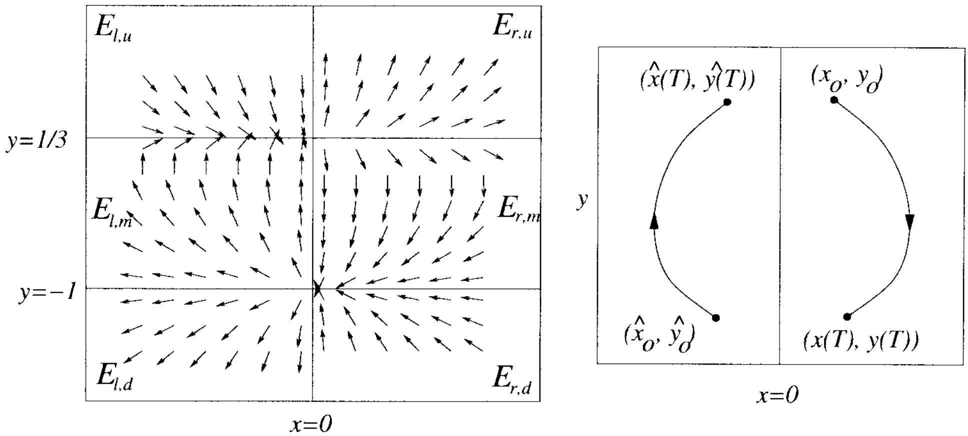

Fig. 2. Normalized vector field and symmetry of trajectories.

Lemma 7.1. *If $(\phi(t), \theta(t))$, $t \in [0, T]$ is a trajectory of the differential equation then*

- *(symmetry) $(\hat{\phi}(t), \hat{\theta}(t))$, $t \in [c, c+T]$ defined by*

$$\begin{cases} \hat{\phi}(t) = -\phi(c+T-t) \\ \hat{\theta}(t) = \theta(c+T-t)) \end{cases}$$

- *and (similarity) $(\tilde{\phi}(\tau), \tilde{\theta}(\tau))$, $\tau \in [c, c+T/\alpha]$, with $\tau = t/\alpha + c$ defined by*

$$\begin{cases} \tilde{\phi}(\tau) = \frac{1}{\alpha}\phi(t) \\ \tilde{\theta}(\tau) = \theta(t), \end{cases}$$

where $\alpha > 0$ and c are two constants, are also two trajectories of the differential equation.

Proof. Straightforward computations show that $(\hat{\phi}(t), \hat{\theta}(t))$, $\forall t \in [c, c+T]$, and $(\tilde{\phi}(\tau), \tilde{\theta}(\tau))$, $\forall \tau \in [c, c+T/\alpha]$ verify the differential equation since $(\phi(t), \theta(t))$, $\forall t \in [0, T]$ verifies it. ∎

Thanks to the symmetry property, it is enough to study the dynamic on the half plane $\phi \geq 0$. This property (together with the similarity property) will also be very useful for the computation of solutions: it will be enough to compute accurately very few trajectories (in fact 3) to get all the others.

Proposition 7.1. *Starting from the initial condition (ϕ_0, θ_0), with $\phi_0 > 0$, the dynamical behavior may be summarized as follows:*

- *if $(\phi_0, \theta_0) \in E_{r,u} = \{\phi > 0\} \times \{\frac{1}{3} < \theta\}$, then the trajectory stays in the set $E_{r,u}$, $\phi(t)$ and $\theta(t)$ are increasing on $[0, +\infty)$ and*

$$\lim_{t \to +\infty} \phi(t) = \lim_{t \to +\infty} \theta(t) = +\infty.$$

- *If* $(\phi_0, \theta_0) \in E_{r,m} = \{\phi > 0\} \times \{-1 < \theta < \frac{1}{3}\}$, *then the trajectory stays in the set* $E_{r,m}$ *until a time* $\bar{t}$ *where* $(\phi(\bar{t}), \theta(\bar{t})) = (0, -1)$ *with*

$$\bar{t} = \int_0^{+\infty} 2\phi_0 e^{\frac{2\xi}{3}} \left| \frac{C-1}{Ce^{4\xi}-1} \right|^{\frac{2}{3}} d\xi, \quad \text{and } C = C(\theta_0) = \frac{\theta_0 - \frac{1}{3}}{\theta_0 + 1}.$$

Moreover, $\theta(t)$ *is decreasing on* $[0, \bar{t}]$ *and starting from* $\theta_0 > 0$, $\phi(t)$ *first increases until a time* $\hat{t}$ *(which corresponds to* $\phi(\hat{t}) = 0$*) where* $\phi(t)$ *begins to decrease.*
- *If* $(\phi_0, \theta_0) \in E_{r,d} = \{\phi > 0\} \times \{\theta < -1\}$, *then the trajectory stays in the set* $E_{r,d}$ *until the time* $\bar{t}$ *where* $(\phi(\bar{t}), \theta(\bar{t})) = (0, -1)$.
- *The points* $(0, \frac{1}{3})$ *and* $(0, 1)$ *are singular for the dynamic: there are an infinite number of trajectories across them. Any trajectory starting from* (ϕ_0, θ_0) *in* $E_{r,m}$, $E_{r,d}$, $E_{l,m}$, *or* $E_{l,u}$ *reaches one of the two singular points but may be extended (after the singularity) in the previous mentioned symmetric way (which allows the maximum regularity for these completed trajectories). In particular, a trajectory starting at* $(\phi_0, \theta_0) \in E_{r,m}$ *may be extended at time* $\bar{t}$ *in* $E_{l,m}$ *(by the* $x = 0$ *axis symmetry). Arriving at* $(0, \frac{1}{3})$ *this second part of the trajectory may also be extended in* $E_{r,m}$ *and finally reaches the point* (ϕ_0, θ_0) *at time* T *(depending on* (ϕ_0, θ_0)*), giving rise to a periodic trajectory. For these periodic orbits, which cut the* $\theta = 0$ *axis, say for* $\phi = \Phi$, *the period* T *is a linear function of* Φ, *more precisely:*

$$T = \frac{2^{4/3}}{\sqrt{3}} \beta\left(\frac{1}{6}, \frac{1}{2}\right) \Phi \simeq 10.6\Phi.$$

- *The integral curves are given by*

$$\phi = \phi_0 \left| \frac{\theta - 1/3}{\theta_0 - 1/3} \right|^{\frac{1}{6}} \left| \frac{\theta + 1}{\theta_0 + 1} \right|^{\frac{1}{2}}. \tag{10}$$

Proof. The main tool consists in applying a time change $(t \mapsto \tau)$ defined by

$$\begin{cases} \frac{d\tau}{dt} = \frac{1}{2\phi} \\ \tau(t=0) = 0 \end{cases} \qquad \text{while } \phi > 0,$$

because we can get the trajectories analytically as a function of τ. The function θ verifies the differential equation

$$\frac{d\theta}{d\tau} = 3\left(\theta - \frac{1}{3}\right)(\theta + 1), \qquad \text{while } \phi > 0,$$

which yields

$$\theta(\tau) = \frac{\frac{1}{3} + Ce^{4\tau}}{1 - Ce^{4\tau}}, \qquad \text{while } \phi > 0 \tag{11}$$

with

$$C = C(\theta_0) = \frac{\theta_0 - \frac{1}{3}}{\theta_0 + 1} \quad \text{and} \quad \begin{cases} C > 1 \text{ when } \theta_0 \in (-\infty, -1) \\ C < 0 \text{ when } \theta_0 \in (-1, \frac{1}{3}) \\ C \in (0, 1) \text{ when } \theta_0 \in (\frac{1}{3}, +\infty) \end{cases}.$$

Starting from $(\phi_0, \theta_0) \in E_{r,u}$, we have

$$\lim_{\tau \to \bar{\tau}-} \theta(\tau) = +\infty, \ \bar{\tau} = -\log(C)/4.$$

We will see later on that in this case the time t goes also to $+\infty$ (as a function of τ) and so there is no blow up at some finite time $\bar{t}$.

The expression of ϕ in terms of τ can be obtained via

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = 2\phi\theta, \qquad \text{while } \phi > 0.$$

We get after some computations

$$\phi(\tau) = \phi_0 e^{\frac{2\tau}{3}} \left| \frac{C-1}{Ce^{4\tau}-1} \right|^{\frac{2}{3}}, \qquad \text{while } \phi > 0. \tag{12}$$

Like for $\theta(\tau)$, we see that starting from $(\phi_0, \theta_0) \in E_{r,u}$ we have

$$\lim_{\tau \to \bar{\tau}-} \phi(\tau) = +\infty, \ \bar{\tau} = -\log(C)/4.$$

Finally, from $dt = 2\phi \, d\tau$ we have the relation giving the time t in terms of τ:

$$t(\tau) = \int_0^\tau 2\phi_0 e^{\frac{2\xi}{3}} \left| \frac{C-1}{Ce^{4\xi}-1} \right|^{\frac{2}{3}} d\xi, \qquad \text{while } \phi > 0. \tag{13}$$

We can now prove the stated behavior:

- if $(\phi_0, \theta_0) \in E_{r,u}$ then the time change $t \mapsto \tau$ is a diffeomorphism from $[0, +\infty)$ to $[0, \bar{\tau})$ $(\bar{\tau} = \log(C)/4)$. Combined with (11) and (12) we have then

$$\lim_{t \to +\infty} \phi(t) = \lim_{t \to +\infty} \theta(t) = +\infty;$$

- if $(\phi_0, \theta_0) \in E_{r,m}$, we get $C < 0$ and the time change is a diffeomorphism from $[0, \bar{t})$ to $[0, +\infty)$, with

$$\bar{t} = \int_0^{+\infty} 2\phi_0 e^{\frac{2\xi}{3}} \left| \frac{C-1}{Ce^{4\xi}-1} \right|^{\frac{2}{3}} d\xi$$

  and (11) and (12) show that

$$\lim_{t \to \bar{t}-} \theta(t) = -1, \ \lim_{t \to \bar{t}-} \phi(t) = 0.$$

  Furthermore, computing derivatives, we see that θ is strictly decreasing on $[0, \bar{t})$ and
  - if $\theta_0 > 0$, then ϕ is increasing on $[0, t(\hat{\tau}))$ and decreasing on $(t(\hat{\tau}), \bar{t}))$, with

$$\hat{\tau} = -\frac{1}{4}\log(-3C),$$

    corresponding to the time when $\theta = 0$.
  - If $\theta_0 \le 0$, $\phi(t)$ is decreasing on $[0, \bar{t})$.

It is obvious that a trajectory starting from $(\phi_0, \theta_0) \in E_{r,u}$ and reaching $(0, -1)$ at time $\bar{t}$ may be completed by a first part connecting $(0, 1/3)$ to (ϕ_0, θ_0) in time $-t(-\infty)$. The family of all these trajectories going from $(0, 1/3)$ to $(0, -1)$ may be parametrized in a unique way by $\Phi > 0$, Φ selecting the only one passing at the point $(\Phi, 0)$. The time to go from $(0, 1/3)$ to $(0, -1)$ is

$$\int_{-\infty}^{+\infty} 2\phi_0 e^{\frac{2\xi}{3}} \left| \frac{C-1}{Ce^{4\xi}-1} \right|^{\frac{2}{3}} d\xi = \frac{2^{1/3}}{\sqrt{3}} \beta | \left(\frac{1}{6}, \frac{1}{2} \right) \Phi. \tag{14}$$

If we complete such a trajectory by the symmetrized trajectory in $E_{l,m}$ we have then a periodic orbit of period $T = 2^{4/3}/\sqrt{3}\,\beta(1/6, 1/2)\Phi$.

- if $(\phi_0, \theta_0) \in E_{r,d}$, we have $C > 1$; the time change $t \mapsto \tau$ is a diffeomorphism from $[0, \bar{t})$ to $[0, +\infty)$. Here ϕ is decreasing while θ is increasing.
- From (11) and (12) we get (10), giving the integral curves independently of t. ▪

Like for all analytical expressions, we can compute safely the trajectories in terms of t. In fact, the only numerical work involved consists of approximating the integral (13), which is easy. By the way, we have also implementeded a (good) numerical differential equation solver (at the very beginning of this study) and we note that different results may be obtained when crossing (near) the singular points, where the solver may continue the trajectory to various ones. This is not surprising!

Figure 3 shows two periodic trajectories, one as an unbroken line passing at $(1, 0)$ and the other as a dotted line passing at $(0.5, 0)$ and computed thanks to the similarity property.

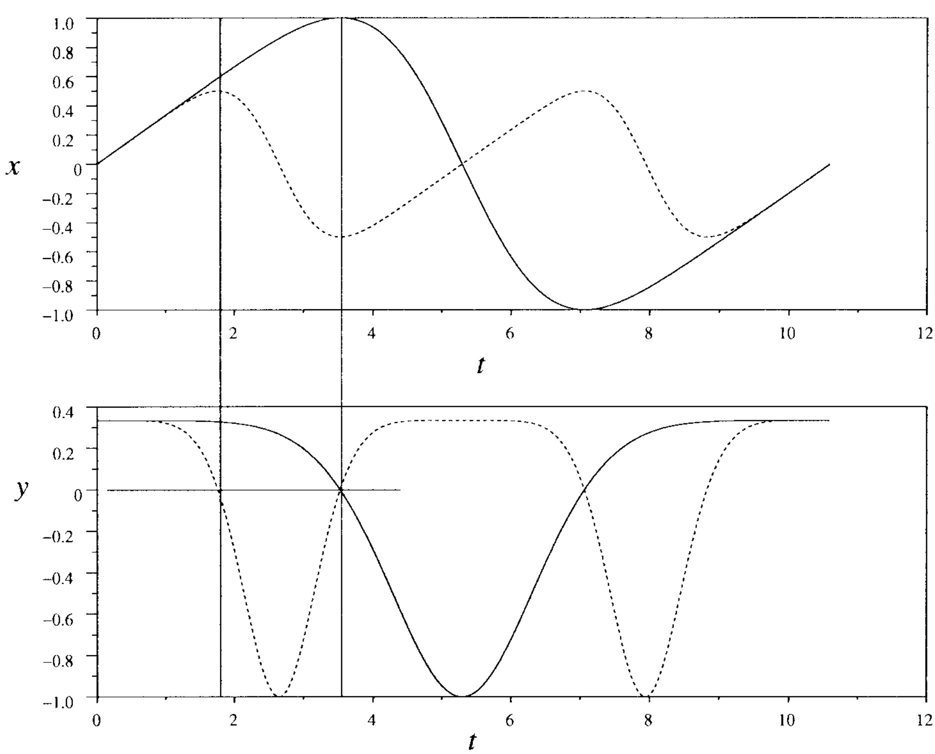

Fig. 3. Two trajectories of the differential equation.

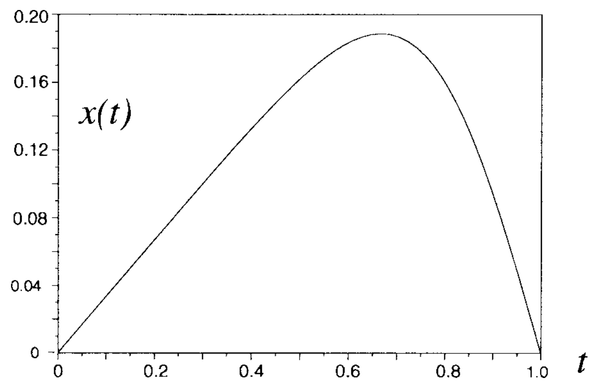

Fig. 4. The positive solution such that $\phi_{II}(0) = \phi_{II}(1) = 0$.

Finally, we are interested in a solution $\phi_{II}(t)$, $t \in [0,1]$ of differential Eq. (9) such that $\phi_{II}(0) = \phi_{II}(1) = 0$ (Fig. 4). If we constrain ϕ_{II} to be positive ($t \in (0,1)$), then this solution is unique and corresponds to the trajectory in $E_{r,m}$, which goes from $(0, 1/3)$ to $(0, -1)$ in a unit time, and so, the one parametrized (see Eq. (14)) by

$$\Phi = \frac{\sqrt{3}}{2^{1/3}\beta\left(\frac{1}{6}, \frac{1}{2}\right)}.$$

Like the period T, the area (i.e., m_{II}) can be computed in term of the β function:

$$m_{II} = \int_0^1 \phi_{II}(t)\,dt = \int_{-\infty}^{+\infty} \phi_{II}(t(\tau))\frac{dt}{d\tau}\,d\tau = \frac{6}{\beta(\frac{1}{6}, \frac{1}{2})^2} \simeq 0.113.$$

8. CONCLUDING REMARKS

We have shown that is possible to analyze dynamic data structures when the distribution on the set of histories is not uniform and when the operations are modeled by time-dependent dynamic random variables. We have recovered some results of Louchard (see [16]) in a more general setting but the temporal inhomogeneity of the dynamic random walk does not allow us to establish a precise large deviation principle from which we could derive the asymptotic distributions of data structure cost functions. Since our stochastic model is very general, it can be applied to a variety of real-world phenomena (including parallel and distributed computing (see [13]), modeling of multi-agents behaviors, and option pricing in financial markets).

APPENDIX

Proof of Lemma 4.1.

1. Existence and Unicity of the Minimizer. The existence of the minimizer is an application of Exercise 4.3.10 in [4]. We use the upper semicontinuity of F and the fact that F is bounded above on any set where the good rate function I is finite. The uniqueness of the minimizer follows from the strict convexity of the functional $I - F$ on its domain.

2. Properties of the Minimizer ϕ_{ll}. The domain of the functional $I - F$ is included in the set of absolutely continuous nonnegative functions vanishing at points 0 and 1. Hence, the minimizer ϕ_{ll} must be an absolutely continuous nonnegative function such that $\phi_{ll}(0) = \phi_{ll}(1) = 0$. Suppose that it is not concave. Define the function $\tilde{\phi}$ by

$$\tilde{\phi}(t) = \sup \left\{ \int_D \dot{\phi}_{ll}(u)\, du \mid D \subset [0,1], |D| = t \right\}.$$

It would verify $I(\phi_{ll}) = I(\tilde{\phi})$ and $F(\phi_{ll}) < F(\tilde{\phi})$. This would contradict the fact that ϕ_{ll} is a minimizer of $I - F$. Hence, the function ϕ_{ll} is a nonzero concave function vanishing at points 0 and 1. This implies that $\dot{\phi}_{ll}(0) > 0$, $\dot{\phi}_{ll}(1) < 0$ and $\phi_{ll} > 0$ on $(0, 1)$.

3. Euler–Lagrange Equation. The link between functional minimization and the Euler–Lagrange equation is not straightforward since we work on the space of absolutely continuous functions. The Euler–Lagrange equation is a second-order differential equation, and we cannot take for granted that ϕ_{ll} is twice differentiable. However, if we suppose that ϕ_{ll} has a continuous second-order derivative, then it satisfies the Euler–Lagrange equation,

$$\frac{d}{dt}\frac{\partial L_{ll}}{\partial y}(\phi, \dot{\phi}) - \frac{\partial L_{ll}}{\partial x}(\phi, \dot{\phi}) = 0, \tag{15}$$

where $L_{ll}(x, y) = \Lambda^*(y) - \log x$ with boundary conditions $\phi(0) = \phi(1) = 0$. This is a standard result from calculus of variations. ∎

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