On the convergence of LePage series in Skorokhod space

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**A B S T R A C T**

We consider the problem of the convergence of the so-called LePage series in the Skorokhod space $D^d = D([0, 1], \mathbb{R}^d)$ and provide a simple criterion based on the moments of the increments of the random process involved in the series. This provides a simple sufficient condition for the existence of an $\alpha$-stable distribution on $\mathbb{R}^d$ with given spectral measure.

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**1. Introduction**

We are interested in the convergence in the Skorokhod space $D^d = D([0, 1], \mathbb{R}^d)$ endowed with the $J_1$-topology of random series of the form

$$X(t) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0, 1],$$

where $\alpha \in (0, 2)$ and:

- $(\Gamma_i)_{i \geq 1}$ is the increasing enumeration of the points of a Poisson point process on $[0, +\infty)$ with Lebesgue intensity;
- $(\varepsilon_i)_{i \geq 1}$ is an i.i.d. sequence of real random variables;
- $(Y_i)_{i \geq 1}$ is an i.i.d. sequence of $D^d$-valued random variables;
- the sequences $(\Gamma_i)$, $(\varepsilon_i)$ and $(Y_i)$ are independent.

Note that a more constructive definition for the sequence $(\Gamma_i)_{i \geq 1}$ is given by

$$\Gamma_i = \sum_{j=1}^{i} \gamma_j, \quad i \geq 1,$$

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where \((y_t)_{t \geq 1}\) is an i.i.d. sequence of random variables with exponential distribution of parameter 1, and independent of \((\epsilon_i)\) and \((Y_i)\).

Series of the form (1) are known as LePage series. For fixed \(t \in [0, 1]\), the convergence in \(\mathbb{R}^d\) of the series (1) is ensured as soon as one of the following conditions is satisfied:

- \(0 < \alpha < 1\), \(\mathbb{E}|\epsilon_1|^\alpha < \infty\) and \(\mathbb{E}|Y_1(t)|^\alpha < \infty\).
- \(1 \leq \alpha < 2\), \(\mathbb{E}|\epsilon_1|^\alpha < \infty\) and \(\mathbb{E}|Y_1(t)|^\alpha < \infty\).

Here \(|\cdot|\) denotes the usual Euclidean norm on \(\mathbb{R}\) or on \(\mathbb{R}^d\). The random variable \(X(t)\) has then an \(\alpha\)-stable distribution on \(\mathbb{R}^d\). Conversely, it is well known that any \(\alpha\)-stable distributions on \(\mathbb{R}^d\) admits a representation in terms of LePage series (see for example Samorodnitsky and Taqqu (1994), Section 3.9).

There is a vast literature on symmetric \(\alpha\)-stable distributions on separable Banach spaces (see e.g. Ledoux and Talagrand (1991), Araujo and Giné (1980)). In particular, any symmetric \(\alpha\)-stable distribution on a separable Banach space can be represented as an almost surely convergent LePage series (see Corollary 5.5 in Ledoux and Talagrand (1991)). The existence of a symmetric \(\alpha\)-stable distribution with a given spectral measure is not automatic and is linked with the notion of stable type of a Banach space; see Theorem 9.27 in Ledoux and Talagrand (1991) for a precise statement. Davydov et al. (2008) consider \(\alpha\)-stable distributions in the more general framework of abstract convex cones.

The space \(\mathbb{D}\) equipped with the norm

\[
\|x\| = \sup\{|x_i(t)|, \ t \in [0, 1], \ i = 1, \ldots, d\}, \quad x = (x_1, \ldots, x_d) \in \mathbb{D},
\]

is a Banach space but is not separable. The uniform topology associated with this norm is finer than the \(J_1\)-topology. On the other hand, the space \(\mathbb{D}\) with the \(J_1\)-topology is Polish, i.e. there exists a metric on \(\mathbb{D}\) compatible with the \(J_1\)-topology that makes \(\mathbb{D}\) a complete and separable metric space. However, such a metric cannot be compatible with the vector space structure since the addition is not continuous in the \(J_1\)-topology. These properties explain why the general theory of stable distributions on separable Banach space cannot be applied to the space \(\mathbb{D}\).

Nevertheless, in the case where the series (1) converges, the distribution of the sum \(X\) defines an \(\alpha\)-stable distribution on \(\mathbb{R}^d\). We can determine the associated spectral measure \(\sigma\) on the unit sphere \(S^d = \{x \in \mathbb{D}; \ |x| = 1\}\). It is given by

\[
\sigma(A) = \frac{\mathbb{E}\left(|\epsilon_1|^\alpha \|Y_1\|^\alpha 1_{\{|\epsilon_1| \leq |Y_1|\}}\right)}{\mathbb{E}(|\epsilon_1|^\alpha \|Y_1\|^\alpha)}, \quad A \in \mathcal{B}(S^d).
\]

This is closely related to regular variations theory (see Hult and Lindskog (2006), Davis and Mikosch (2008)). For all \(r > 0\) and \(A \in \mathcal{B}(S^d)\) such that \(\sigma(\partial A) = 0\), it holds that

\[
\lim_{n \to \infty} n^2 \mathbb{P}\left(\frac{|X|}{\|X\|} \in A \mid \|X\| > rb_n\right) = r^{-\alpha} \sigma(A),
\]

with

\[
b_n = \inf\{r > 0; \ \mathbb{P}(\|X\| < r) \leq n^{-1}\}, \quad n \geq 1.
\]

The random variable \(X\) is said to be regularly varying in \(\mathbb{D}\) with index \(\alpha\) and spectral measure \(\sigma\).

In this framework, convergence of the LePage series (1) in \(\mathbb{D}\) is known in some particular cases only:

- When \(0 < \alpha < 1\), \(\mathbb{E}|\epsilon_1|^\alpha < \infty\) and \(\mathbb{E}\|Y_1\|^\alpha < \infty\), the series (1) converges almost surely uniformly in [0, 1] (see example 4.2 in Davis and Mikosch (2008)).
- When \(1 \leq \alpha < 2\), the distribution of the \(\epsilon_i\)'s is symmetric, \(\mathbb{E}|\epsilon_1|^\alpha < \infty\) and \(Y_i(t) = 1_{[0,1]}(U)\) with \((U_i)_{i \geq 1}\) an i.i.d. sequence of random variables with uniform distribution on [0, 1], the series (1) converges almost surely uniformly on [0, 1] and the limit process \(X\) is a symmetric \(\alpha\)-stable Lévy process (see Rosiński (2001)).

The purpose of this note is to complete these results and to provide a general criterion for almost sure convergence in \(\mathbb{D}\) of the random series (1). Our main result is the following:

**Theorem 1.** Suppose that \(1 \leq \alpha < 2\),

\[
\mathbb{E}\epsilon_1 = 0, \quad \mathbb{E}|\epsilon_1|^\alpha < \infty \quad \text{and} \quad \mathbb{E}\|Y_1\|^\alpha < \infty.
\]

Suppose furthermore that there exist \(\beta_1, \beta_2 > \frac{1}{2}\) and \(F_1, F_2\) nondecreasing continuous functions on [0, 1] such that, for all \(0 \leq t_1 \leq t \leq t_2 \leq 1\),

\[
\mathbb{E}|Y_i(t_2) - Y_i(t_1)|^2 \leq |F_1(t_2) - F_1(t_1)|^{\beta_1},
\]

\[
\mathbb{E}|Y_i(t_2) - Y_i(t_1)|^2 |Y_1(t) - Y_1(t_1)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}.
\]

Then, the LePage series (1) converges almost surely in \(\mathbb{D}\).

The proof of this theorem is detailed in the next section. We provide hereafter a few cases where Theorem 1 can be applied.
Example 1. The example considered by Davis and Mikosch (2008) follows easily from Theorem 1: let $U$ be a random variable with uniform distribution on $[0, 1]$ and consider $Y_1(t) = 1_{[0,t]}(U)$, $t \in [0, 1]$. Then, for $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$
\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = t_2 - t_1 \quad \text{and} \quad \mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 = 0,
$$

so conditions (2) and (3) are satisfied.

Example 2. Example 1 can be generalized in the following way: let $p \geq 1$, $(U_i)_{1 \leq i \leq p}$ independent random variables on $[0, 1]$ and $(R_i)_{1 \leq i \leq p}$ random variables on $\mathbb{R}^d$. Consider

$$
Y_1(t) = \sum_{i=1}^{p} R_i 1_{[0,t]}(U_i).
$$

Assume that for each $i \in \{1, \ldots, p\}$, the cumulative distribution function $F_i$ of $U_i$ is continuous on $[0, 1]$. Assume furthermore that there is some $M > 0$ such that for all $i \in \{1, \ldots, p\}$

$$
\mathbb{E}[R_i^4 \mid \mathcal{F}_U] \leq M \quad \text{almost surely},
$$

where $\mathcal{F}_U = \sigma(U_1, \ldots, U_p)$. This is for example the case when the $R_i$’s are uniformly bounded by $M^{1/4}$ or when the $R_i$’s have finite fourth moment and are independent of the $U_i$’s. Simple computations entail that under condition (4), it holds for all $0 \leq t_1 \leq t \leq t_2 \leq 1$ that

$$
\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 \leq M^{1/2} p^2 |F(t_2) - F(t_1)|^2
$$

and

$$
\mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 \leq Mp^4 |F(t_2) - F(t_1)|^4.
$$

with $F(t) = \sum_{i=1}^{p} F_i(t)$. So conditions (2) and (3) are satisfied and Theorem 1 can be applied in this case.

Example 3. A further natural example is the case where $Y_1(t)$ is a Poisson process with intensity $\lambda > 0$ on $[0, 1]$. Then, for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$
\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = \lambda |t_2 - t_1| + \lambda^2 |t_2 - t_1|^2
$$

and

$$
\mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 = (\lambda |t_2 - t| + \lambda^2 |t_2 - t|)(\lambda |t - t_1| + \lambda^2 |t - t_1|)^2
$$

and we easily see that conditions (2) and (3) are satisfied.

2. Proof

For the sake of clarity, we divide the proof of Theorem 1 into five steps.

Step 1. According to Lemma 1.5.1 in Samorodnitsky and Taqqu (1994), it holds almost surely that for $k$ large enough,

$$
|I_k^{-1/\alpha} - k^{-1/\alpha}| \leq 2\alpha^{-1} k^{-1/\alpha} \left(\frac{\ln \ln k}{k}\right). \tag{5}
$$

This implies the a.s. convergence of the series

$$
\sum_{i=1}^{\infty} |I_i^{-1/\alpha} - i^{-1/\alpha}| \|\varepsilon_i\| Y_i \| < \infty. \tag{6}
$$

The series (6) does indeed have nonnegative terms, and (5) implies that the following conditional expectation is finite:

$$
\mathbb{E} \left[ \sum_{i=1}^{\infty} |I_i^{-1/\alpha} - i^{-1/\alpha}| \|\varepsilon_i\| Y_i \mid \mathcal{F}_T \right] = \mathbb{E}[\varepsilon_1] \mathbb{E}[Y_1] \sum_{i=1}^{\infty} |I_i^{-1/\alpha} - i^{-1/\alpha}|
$$

where $\mathcal{F}_T = \sigma(I_i, i \geq 1)$.

This proves that (6) holds true and it is enough to prove the a.s. convergence in $\mathbb{D}^d$ of the series

$$
Z(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0, 1], \tag{7}
$$

Step 2. Next, consider

$$
\tilde{Z}(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \tilde{\varepsilon}_i Y_i(t), \quad t \in [0, 1]. \tag{8}
$$
with
\[ \tilde{e}_i = \varepsilon_i 1_{[\varepsilon_i^\alpha > i]}, \quad i \geq 1. \]

We prove that the series (7) and (8) differ only by a finite number of terms. We have indeed
\[ \sum_{i=1}^\infty \mathbb{P} (\tilde{e}_i \neq e_i) = \sum_{i=1}^\infty \mathbb{P} (|\varepsilon_i^\alpha > i) \leq \mathbb{E}|\varepsilon_1|^\alpha < \infty \]
and the Borel–Cantelli lemma implies that almost surely \( \tilde{e}_i = e_i \) for \( i \) large enough. So, the two series (7) and (8) have the same nature and it is enough to prove the convergence in \( D^d \) of the series (8).

**Step 3.** As a preliminary for step 4, we prove several estimates involving the moments of the random variables \( (\tilde{e}_i)_{i \geq 1} \). First, for all \( m > \alpha \),

\[ C(\alpha, m) := \sum_{i=1}^\infty i^{-m/\alpha} \mathbb{E}(|\tilde{e}_i|^m) < \infty. \quad (9) \]

We have indeed
\[ C(\alpha, m) = \sum_{i=1}^\infty i^{-m/\alpha} \mathbb{E}(|\varepsilon_1|^{m} 1_{[|\varepsilon_1|^{\alpha/\alpha} \leq i^{1/\alpha}}) \]
\[ = \mathbb{E}(|\varepsilon_1|^{m} \sum_{i=1}^\infty i^{-m/\alpha} 1_{[|\varepsilon_1|^{\alpha/\alpha} \leq i^{1/\alpha}}) \]
\[ \leq CE(|\varepsilon_1|^{m} |\varepsilon_1|^{\alpha-m}) = CE(|\varepsilon_1|^{\alpha}) < \infty \]
where the constant \( C = \sup_{x>0} x^{m/\alpha-1} \sum_{|x|} i^{-m/\alpha} \) is finite since \( m > \alpha \)
\[ \lim_{x \to \infty} x^{m/\alpha-1} \sum_{i \geq x} i^{-m/\alpha} = \frac{\alpha}{m - \alpha}. \]

Similarly, we also have
\[ C(\alpha, 1) := \sum_{i=1}^\infty i^{-1/\alpha} |\mathbb{E}(\tilde{e}_i)| < \infty. \quad (10) \]

Indeed, the assumption \( \mathbb{E}\varepsilon_1 = 0 \) implies \( \mathbb{E}(\tilde{e}_i) = \mathbb{E}(\varepsilon_1 1_{[|\varepsilon_1|^{\alpha/\alpha} > i^{1/\alpha}}) \). Hence,
\[ \sum_{i=1}^\infty i^{-1/\alpha} |\mathbb{E}(\tilde{e}_i)| \leq \sum_{i=1}^\infty i^{-1/\alpha} \mathbb{E}(|\varepsilon_1| 1_{[|\varepsilon_1|^{\alpha/\alpha} > i^{1/\alpha}}) \]
\[ = \mathbb{E}(|\varepsilon_1| \sum_{i=1}^\infty i^{-1/\alpha}) \]
\[ \leq \mathbb{E}(|\varepsilon_1| C(|\varepsilon_1|^{\alpha})^{1-1/\alpha}) = C |\varepsilon_1|^\alpha < \infty \]
where the constant \( C' = \sup_{x>0} x^{1/\alpha-1} \sum_{|x|} i^{-1/\alpha} \) is finite.

**Step 4.** For \( n \geq 1 \), consider the partial sum
\[ \tilde{Z}_n(t) = \sum_{i=1}^n i^{-1/\alpha} \tilde{e}_i Y_i(t), \quad t \in [0, 1]. \quad (11) \]

We prove that the sequence of processes \( (\tilde{Z}_n)_{n \geq 1} \) is tight in \( D^d \). Following Theorem 3 in Gikhman and Skorokhod (2004), Chapter 6, Section 3, it is enough to show that there exists \( \beta > 1/2 \) and a nondecreasing continuous function \( F \) on \( [0, 1] \) such that
\[ \mathbb{E} (\tilde{Z}_n(t_2) - \tilde{Z}_n(t_1))^2 |\tilde{Z}_n(t) - \tilde{Z}_n(t_1)|^2 \leq |F(t_2) - F(t_1)|^{2\beta}, \]
for all \( 0 \leq t_1 \leq t \leq t_2 \leq 1 \). Remark that in Gikhman and Skorokhod (2004), the result is stated only for \( F(t) \equiv t \). However, the case of a general continuous nondecreasing function \( F \) follows easily from a simple change of variable.
We use the notation $Y(t) = (Y^p(t))_{1 \leq p \leq d}$. If $\mathbb{I} \subseteq \mathbb{N}$ and $i = (i_1, i_2, i_3, i_4) \in \mathbb{I}$, we have

$$
\mathbb{E}|\tilde{Z}_n(t_2) - \tilde{Z}_n(t)|^2 \leq \mathbb{E} \left| \sum_{j=1}^n j^{-\alpha/\alpha} \tilde{E}_j(Y(t) - Y(t)) \right|^2 \leq \sum_{j=1}^n j^{-\alpha/\alpha} \mathbb{E} \left| \tilde{E}_j(Y(t) - Y(t)) \right|^2 \leq \sum_{i=1}^n \frac{(i_1 i_2 i_3 i_4)^{1/\alpha}}{|i|^{1/\alpha}} \mathbb{E} \left| \tilde{E}_1 \tilde{E}_1 \tilde{E}_1 \tilde{E}_1 \right|^2 \mathbb{E} \left| Y(t) - Y(t) \right|^2 \tag{13}
$$

Consider $\sim_i$, the equivalence relation on $\{1, \ldots, 4\}$ defined by

$$
j \sim_i j \text{ if and only if } j = i_p.
$$

Let $\mathcal{P}$ be the set of all partitions of $\{1, \ldots, 4\}$ and $\tau(\mathbf{i})$ be the partition of $\{1, 2, 3, 4\}$ given by the equivalence classes of $\sim_i$. We introduce these definitions because, since the $Y_i$'s are i.i.d., the term $D_i(t, t_1, t_2)$ depends on $\mathbf{i}$ only through the associated partition $\tau(\mathbf{i})$. For example, if $\tau(\mathbf{i}) = \{1, 2, 3, 4\}$, i.e. if $i_1 = i_2 = i_3 = i_4$, then

$$
D_i(t, t_1, t_2) = \mathbb{E}|Y(t_1) - Y(t_1)|^2 |Y_i(t_2) - Y_i(t)|^2.
$$

Or if $\tau(\mathbf{i}) = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, i.e. if the indices $i_1, \ldots, i_4$ are pairwise distinct, then

$$
D_i(t, t_1, t_2) = \mathbb{E}|Y(t_1) - Y(t_1)| \mathbb{E}|Y_i(t_2) - Y_i(t)|^2.
$$

For $\tau \in \mathcal{P}$, we denote by $D_\tau(t_1, t_2)$ the common value of the terms $D_i(t, t_1, t_2)$ corresponding to indices $\mathbf{i}$ such that $\tau(\mathbf{i}) = \tau$. Define also

$$
S_{n, \tau} = \sum_{\mathbf{i} \in \{1, \ldots, n\}^4, \tau(\mathbf{i}) = \tau} (i_1 i_2 i_3 i_4)^{1/\alpha} |\mathbb{E}(\tilde{E}_1 \tilde{E}_1 \tilde{E}_1 \tilde{E}_1)|.
$$

With this notation, Eq. (15) can be rewritten as

$$
\mathbb{E}|\tilde{Z}_n(t_2) - \tilde{Z}_n(t)|^2 \leq d^2 \sum_{\tau \in \mathcal{P}} S_{n, \tau} D_\tau(t, t_1, t_2). \tag{16}
$$

Under conditions (2) and (3), we will prove that for each $\tau \in \mathcal{P}$, there exist $\beta_\tau > 1/2$, a nondecreasing continuous function $F_\tau$ on $[0, 1]$ and a constant $S_\tau > 0$ such that

$$
D_\tau(t, t_1, t_2) \leq |F_\tau(t) - F_\tau(t)|^{2\beta_\tau}, \quad 0 \leq t_1 \leq t \leq t_2, \tag{17}
$$

and

$$
S_{n, \tau} \leq S_\tau, \quad n \geq 1. \tag{18}
$$

Eqs. (16)–(18) together imply inequality (12) for some suitable choices of $\beta > 1/2$ and $F$.

It remains to prove inequalities (17) and (18). If $\tau = \{1, 2, 3, 4\}$,

$$
D_\tau(t, t_1, t_2) \leq \mathbb{E}|Y(t) - Y(t)|^2 |Y(t_2) - Y(t)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}
$$

and

$$
S_2^4 \leq \sum_{i=1}^n i^{-4/\alpha} \mathbb{E} i_4^4 \leq C(\alpha, 4).
$$

If $\tau = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, the Cauchy–Schwartz inequality entails

$$
D_\tau(t, t_1, t_2) \leq \mathbb{E}|Y(t) - Y(t)| \mathbb{E}|Y(t_2) - Y(t)| \leq |F_1(t_2) - F_1(t_1)|^{2\beta_1}
$$

and

$$
S_1^4 \leq \sum_{\mathbf{i} \in \{1, \ldots, n\}^4, \tau(\mathbf{i}) = \tau} (i_1 i_2 i_3 i_4)^{1/\alpha} \mathbb{E} \tilde{E}_1 \mathbb{E} \tilde{E}_1 \mathbb{E} \tilde{E}_1 \mathbb{E} \tilde{E}_1 \leq C(\alpha, 1)^4.
$$
Similarly, for \( \tau = \{1, 2, 3\} \cup \{4\}, \)
\[
D_\tau(t, t_1, t_2) = \mathbb{E}[|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)|]
\leq |F_1(t) - F_1(t_1)|^\beta_1 |F_2(t_2) - F_2(t_1)|^\beta_2 |F_1(t_2) - F_1(t)|^\beta_1
\leq |(F_1 + F_2)(t_2) - (F_1 + F_2)(t_1)|^{\beta_1+\beta_2}
\]
and
\[
S_n^\tau \leq \sum_{1 \leq j \neq k \leq n} (t^j)^{-1/\alpha} \mathbb{E}|\tilde{\xi}_j|^2 |\mathbb{E}\tilde{\xi}_j| \leq C(\alpha, 3) C(\alpha, 3).
\]
or for \( \tau = \{1, 2\} \cup \{3\} \cup \{4\}, \)
\[
D_\tau(t, t_1, t_2) = \mathbb{E}[|Y_1(t) - Y_1(t_1)|^2 (\mathbb{E}|Y_1(t_2) - Y_1(t)|)^2]
\leq |F_1(t) - F_1(t_1)|^\beta_1 |F_1(t_2) - F_1(t)|^\beta_1
\leq |F_1(t_2) - F_1(t_1)|^{\beta_1}
\]
and
\[
S_n^\tau \leq \sum_{1 \leq j \neq k \leq n} (t^j k)^{-1/\alpha} \mathbb{E}|\tilde{\xi}_j|^2 |\mathbb{E}\tilde{\xi}_j| |\mathbb{E}\tilde{\xi}_k| \leq C(\alpha, 2) C(\alpha, 1)^2.
\]

Similar computations can be checked in all remaining cases. The cardinality of \( \mathcal{P} \) is equal to 13.

Step 5. We prove Theorem 1. For each fixed \( t \in [0, 1] \), Kolmogorov’s three-series theorem implies that \( \tilde{Z}_n(t) \) converges almost surely as \( n \to \infty \). So the finite-dimensional distributions of \( (\tilde{Z}_n)_{n \geq 1} \) converge. The tightness in \( \mathbb{D}^d \) of the sequence has already been proved in step 4, so \( (\tilde{Z}_n)_{n \geq 1} \) weakly converges in \( \mathbb{D}^d \) as \( n \to \infty \). We then apply Theorem 1 of Kallenberg (1974) and deduce that \( \tilde{Z}_n \) converges almost surely in \( \mathbb{D}^d \). In view of step 1 and step 2, this yields the almost sure convergence of the series (1). \( \square \)

References


