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On the convergence of LePage series in Skorokhod space

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ABSTRACT

We consider the problem of the convergence of the so-called LePage series in the Skorokhod space $\mathbb{D}^d = \mathbb{D}([0, 1], \mathbb{R}^d)$ and provide a simple criterion based on the moments of the increments of the random process involved in the series. This provides a simple sufficient condition for the existence of an α-stable distribution on $\mathbb{D}^d$ with given spectral measure.

1. Introduction

We are interested in the convergence in the Skorokhod space $\mathbb{D}^d = \mathbb{D}([0, 1], \mathbb{R}^d)$ endowed with the J_1-topology of random series of the form

$$X(t) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0, 1], \tag{1}$$

where $\alpha \in (0, 2)$ and:

- $(\Gamma_i)_{i\geq 1}$ is the increasing enumeration of the points of a Poisson point process on $[0, +\infty)$ with Lebesgue intensity;
- $(\varepsilon_i)_{i\geq 1}$ is an i.i.d. sequence of real random variables;
- $(Y_i)_{i\geq 1}$ is an i.i.d. sequence of $\mathbb{D}^d$-valued random variables;
- the sequences (Γ_i), (ε_i) and (Y_i) are independent.

Note that a more constructive definition for the sequence $(\Gamma_i)_{i\geq 1}$ is given by

$$\Gamma_i = \sum_{j=1}^{i} \gamma_j, \quad i \geq 1,$$

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where $(\gamma_i)_{i\geq 1}$ is an i.i.d. sequence of random variables with exponential distribution of parameter 1, and independent of (ε_i) and (Y_i).

Series of the form (1) are known as LePage series. For fixed $t \in [0,1]$, the convergence in $\mathbb{R}^d$ of the series (1) is ensured as soon as one of the following conditions is satisfied:

- $0 < \alpha < 1$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$,
- $1 \leq \alpha < 2$, $\mathbb{E}\varepsilon_1 = 0$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$.

Here $|.|$ denotes the usual Euclidean norm on $\mathbb{R}$ or on $\mathbb{R}^d$. The random variable $X(t)$ has then an α-stable distribution on $\mathbb{R}^d$. Conversely, it is well known that any α-stable distributions on $\mathbb{R}^d$ admits a representation in terms of LePage series (see for example Samorodnitsky and Taqqu (1994), Section 3.9).

There is a vast literature on symmetric α-stable distributions on separable Banach spaces (see e.g. Ledoux and Talagrand (1991), Araujo and Giné (1980)). In particular, any symmetric α-stable distribution on a separable Banach space can be represented as an almost surely convergent LePage series (see Corollary 5.5 in Ledoux and Talagrand (1991)). The existence of a symmetric α-stable distribution with a given spectral measure is not automatic and is linked with the notion of stable type of a Banach space; see Theorem 9.27 in Ledoux and Talagrand (1991) for a precise statement. Davydov et al. (2008) consider α-stable distributions in the more general framework of abstract convex cones.

The space $\mathbb{D}^d$ equipped with the norm

$$\|x\| = \sup\{|x_i(t)|,\ t \in [0,1],\ i = 1,\dots,d\}, \quad x = (x_1,\dots,x_d) \in \mathbb{D}^d,$$

is a Banach space but is not separable. The uniform topology associated with this norm is finer than the J_1-topology. On the other hand, the space $\mathbb{D}^d$ with the J_1-topology is Polish, i.e. there exists a metric on $\mathbb{D}^d$ compatible with the J_1-topology that makes $\mathbb{D}^d$ a complete and separable metric space. However, such a metric cannot be compatible with the vector space structure since the addition is not continuous in the J_1-topology. These properties explain why the general theory of stable distributions on separable Banach space cannot be applied to the space $\mathbb{D}^d$.

Nevertheless, in the case where the series (1) converges, the distribution of the sum X defines an α-stable distribution on $\mathbb{D}^d$. We can determine the associated spectral measure σ on the unit sphere $\mathbb{S}^d = \{x \in \mathbb{D}^d;\ \|x\| = 1\}$. It is given by

$$\sigma(A) = \frac{\mathbb{E}\Big(|\varepsilon_1|^\alpha \|Y_1\|^\alpha \mathbf{1}_{\{\mathrm{sign}(\varepsilon_1)Y_1/\|Y_1\| \in A\}}\Big)}{\mathbb{E}(|\varepsilon_1|^\alpha \|Y_1\|^\alpha)}, \quad A \in \mathcal{B}(\mathbb{S}^d).$$

This is closely related to regular variations theory (see Hult and Lindskog (2006), Davis and Mikosch (2008)). For all $r > 0$ and $A \in \mathcal{B}(\mathbb{S}^d)$ such that $\sigma(\partial A) = 0$, it holds that

$$\lim_{n\to\infty} n\mathbb{P}\left(\frac{X}{\|X\|} \in A \mid \|X\| > rb_n\right) = r^{-\alpha}\sigma(A),$$

with

$$b_n = \inf\{r > 0;\ \mathbb{P}(\|X\| < r) \leq n^{-1}\}, \quad n \geq 1.$$

The random variable X is said to be regularly varying in $\mathbb{D}^d$ with index α and spectral measure σ.

In this framework, convergence of the LePage series (1) in $\mathbb{D}^d$ is known in some particular cases only:

- When $0 < \alpha < 1$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}\|Y_1\|^\alpha < \infty$, the series (1) converges almost surely uniformly in $[0,1]$ (see example 4.2 in Davis and Mikosch (2008)).
- When $1 \leq \alpha < 2$, the distribution of the ε_i's is symmetric, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $Y_i(t) = \mathbf{1}_{[0,t]}(U)$ with $(U_i)_{i\geq 1}$ an i.i.d. sequence of random variables with uniform distribution on $[0,1]$, the series (1) converges almost surely uniformly on $[0,1]$ and the limit process X is a symmetric α-stable Lévy process (see Rosiński (2001)).

The purpose of this note is to complete these results and to provide a general criterion for almost sure convergence in $\mathbb{D}^d$ of the random series (1). Our main result is the following:

Theorem 1. *Suppose that* $1 \leq \alpha < 2$,

$$\mathbb{E}\varepsilon_1 = 0, \qquad \mathbb{E}|\varepsilon_1|^\alpha < \infty \quad \text{and} \quad \mathbb{E}\|Y_1\|^\alpha < \infty.$$

Suppose furthermore that there exist $\beta_1, \beta_2 > \frac{1}{2}$ *and* F_1, F_2 *nondecreasing continuous functions on* $[0,1]$ *such that, for all* $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}|Y_1(t_2) - Y_1(t_1)|^2 \leq |F_1(t_2) - F_1(t_1)|^{\beta_1}, \tag{2}$$

$$\mathbb{E}|Y_1(t_2) - Y_1(t)|^2 |Y_1(t) - Y_1(t_1)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}. \tag{3}$$

Then, the LePage series (1) converges almost surely in $\mathbb{D}^d$.

The proof of this theorem is detailed in the next section. We provide hereafter a few cases where Theorem 1 can be applied.

Example 1. The example considered by Davis and Mikosch (2008) follows easily from Theorem 1: let U be a random variable with uniform distribution on $[0, 1]$ and consider $Y_1(t) = \mathbf{1}_{[0,t]}(U)$, $t \in [0, 1]$. Then, for $0 \le t_1 \le t \le t_2 \le 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = t_2 - t_1 \quad \text{and} \quad \mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 = 0,$$

so conditions (2) and (3) are satisfied.

Example 2. Example 1 can be generalized in the following way: let $p \ge 1$, $(U_i)_{1\le i\le p}$ independent random variables on $[0, 1]$ and $(R_i)_{1\le i\le p}$ random variables on $\mathbb{R}^d$. Consider

$$Y_1(t) = \sum_{i=1}^{p} R_i \mathbf{1}_{[0,t]}(U_i).$$

Assume that for each $i \in \{1, \ldots, p\}$, the cumulative distribution function F_i of U_i is continuous on $[0, 1]$. Assume furthermore that there is some $M > 0$ such that for all $i \in \{1, \ldots, p\}$

$$\mathbb{E}[R_i^4 \mid \mathcal{F}_U] \le M \quad \text{almost surely,} \tag{4}$$

where $\mathcal{F}_U = \sigma(U_1, \ldots, U_p)$. This is for example the case when the R_i's are uniformly bounded by $M^{1/4}$ or when the R_i's have finite fourth moment and are independent of the U_i's. Simple computations entail that under condition (4), it holds for all $0 \le t_1 \le t \le t_2 \le 1$ that

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 \le M^{1/2} p^2 |F(t_2) - F(t_1)|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 \le M p^4 |F(t_2) - F(t_1)|^4.$$

with $F(t) = \sum_{i=1}^{p} F_i(t)$. So conditions (2) and (3) are satisfied and Theorem 1 can be applied in this case.

Example 3. A further natural example is the case where $Y_1(t)$ is a Poisson process with intensity $\lambda > 0$ on $[0, 1]$. Then, for all $0 \le t_1 \le t \le t_2 \le 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = \lambda|t_2 - t_1| + \lambda^2|t_2 - t_1|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 = (\lambda|t_2 - t| + \lambda^2|t_2 - t|^2)(\lambda|t - t_1| + \lambda^2|t - t_1|^2)$$

and we easily see that conditions (2) and (3) are satisfied.

2. Proof

For the sake of clarity, we divide the proof of Theorem 1 into five steps.

Step 1. According to Lemma 1.5.1 in Samorodnitsky and Taqqu (1994), it holds almost surely that for k large enough,

$$|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}| \le 2\alpha^{-1} k^{-1/\alpha} \sqrt{\frac{\ln \ln k}{k}}. \tag{5}$$

This implies the a.s. convergence of the series

$$\sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}|\, |\varepsilon_i|\, \|Y_i\| < \infty. \tag{6}$$

The series (6) does indeed have nonnegative terms, and (5) implies that the following conditional expectation is finite:

$$\mathbb{E}\left[\sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}|\, |\varepsilon_i|\, \|Y_i\| \,\middle|\, \mathcal{F}_\Gamma \right] = \mathbb{E}|\varepsilon_1|\, \mathbb{E}\|Y_i\| \sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}|$$

where $\mathcal{F}_\Gamma = \sigma(\Gamma_i, i \ge 1)$.

This proves that (6) holds true and it is enough to prove the a.s. convergence in $\mathbb{D}^d$ of the series

$$Z(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0, 1], \tag{7}$$

Step 2. Next, consider

$$\widetilde{Z}(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \tilde{\varepsilon}_i Y_i(t), \quad t \in [0, 1]. \tag{8}$$

with

$$\tilde{\varepsilon}_i = \varepsilon_i \mathbf{1}_{\{|\varepsilon_i|^\alpha \le i\}}, \quad i \ge 1.$$

We prove that the series (7) and (8) differ only by a finite number of terms. We have indeed

$$\sum_{i=1}^{\infty} \mathbb{P}(\tilde{\varepsilon}_i \neq \varepsilon_i) = \sum_{i=1}^{\infty} \mathbb{P}(|\varepsilon_i|^\alpha > i) \le \mathbb{E}|\varepsilon_1|^\alpha < \infty$$

and the Borel–Cantelli lemma implies that almost surely $\tilde{\varepsilon}_i = \varepsilon_i$ for i large enough. So, the two series (7) and (8) have the same nature and it is enough to prove the convergence in $\mathbb{D}^d$ of the series (8).

Step 3. As a preliminary for step 4, we prove several estimates involving the moments of the random variables $(\tilde{\varepsilon}_i)_{i\ge 1}$. First, for all $m > \alpha$,

$$C(\alpha, m) := \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbb{E}(|\tilde{\varepsilon}_i|^m) < \infty. \tag{9}$$

We have indeed

$$\begin{aligned} C(\alpha, m) &= \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbb{E}(|\varepsilon_i|^m \mathbf{1}_{\{|\varepsilon_i| \le i^{1/\alpha}\}}) \\ &= \mathbb{E}\left(|\varepsilon_1|^m \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbf{1}_{\{i \ge |\varepsilon_1|^\alpha\}} \right) \\ &\le C\mathbb{E}(|\varepsilon_1|^m |\varepsilon_1|^{\alpha-m}) = C\mathbb{E}(|\varepsilon_1|^\alpha) < \infty \end{aligned}$$

where the constant $C = \sup_{x>0} x^{m/\alpha-1} \sum_{i\ge x} i^{-m/\alpha}$ is finite since for $m > \alpha$

$$\lim_{x\to\infty} x^{m/\alpha-1} \sum_{i\ge x}^{\infty} i^{-m/\alpha} = \frac{\alpha}{m-\alpha}.$$

Similarly, we also have

$$C(\alpha, 1) := \sum_{i=1}^{\infty} i^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_i)| < \infty. \tag{10}$$

Indeed, the assumption $\mathbb{E}\varepsilon_i = 0$ implies $\mathbb{E}(\tilde{\varepsilon}_i) = \mathbb{E}(\varepsilon_i \mathbf{1}_{\{|\varepsilon_i|^\alpha > i\}})$. Hence,

$$\begin{aligned} \sum_{i=1}^{\infty} i^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_i)| &\le \sum_{i=1}^{\infty} i^{-1/\alpha} \mathbb{E}(|\varepsilon_1| \mathbf{1}_{\{|\varepsilon_1| > i^{1/\alpha}\}}) \\ &= \mathbb{E}\Big(|\varepsilon_1| \sum_{i=1}^{[|\varepsilon_1|^\alpha]} i^{-1/\alpha} \Big) \\ &\le \mathbb{E}\Big(|\varepsilon_1| C'(|\varepsilon_1|^\alpha)^{1-1/\alpha} \Big) = C'\mathbb{E}|\varepsilon_1|^\alpha < \infty \end{aligned}$$

where the constant $C' = \sup_{x>0} x^{1/\alpha-1} \sum_{i=1}^{[x]} i^{-1/\alpha}$ is finite.

Step 4. For $n \ge 1$, consider the partial sum

$$\widetilde{Z}_n(t) = \sum_{i=1}^{n} i^{-1/\alpha} \tilde{\varepsilon}_i Y_i(t), \quad t \in [0, 1]. \tag{11}$$

We prove that the sequence of processes $(\widetilde{Z}_n)_{n\ge 1}$ is tight in $\mathbb{D}^d$. Following Theorem 3 in Gikhman and Skorokhod (2004), Chapter 6, Section 3, it is enough to show that there exists $\beta > 1/2$ and a nondecreasing continuous function F on [0, 1] such that

$$\mathbb{E}|\widetilde{Z}_n(t_2) - \widetilde{Z}_n(t)|^2 |\widetilde{Z}_n(t) - \widetilde{Z}_n(t_1)|^2 \le |F(t_2) - F(t_1)|^{2\beta}, \tag{12}$$

for all $0 \le t_1 \le t \le t_2 \le 1$. Remark that in Gikhman and Skorokhod (2004), the result is stated only for $F(t) \equiv t$. However, the case of a general continuous nondecreasing function F follows easily from a simple change of variable.

We use the notation $Y(t) = (Y^p(t))_{1\le p\le d}$, $[\![1,n]\!] = \{1,\dots,n\}$ and $\mathbf{i} = (i_1,i_2,i_3,i_4) \in [\![1,n]\!]^4$. We have

$$\mathbb{E}|\widetilde{Z}_n(t_2) - \widetilde{Z}_n(t)|^2|\widetilde{Z}_n(t) - \widetilde{Z}_n(t_1)|^2 = \mathbb{E}\left|\sum_{i=1}^n i^{-1/\alpha}\tilde{\varepsilon}_i(Y_i(t) - Y_i(t_1))\right|^2 \left|\sum_{j=1}^n j^{-1/\alpha}\tilde{\varepsilon}_j(Y_j(t_2) - Y_j(t))\right|^2$$

$$= \sum_{1\le p,q\le d}\sum_{\mathbf{i}\in[[1,n]]^4}(i_1i_2i_3i_4)^{-1/\alpha}\mathbb{E}(\tilde{\varepsilon}_{i_1}\tilde{\varepsilon}_{i_2}\tilde{\varepsilon}_{i_3}\tilde{\varepsilon}_{i_4})\mathbb{E}[(Y^p_{i_1}(t) - Y^p_{i_1}(t_1)) \tag{13}$$

$$(Y^p_{i_2}(t) - Y^p_{i_2}(t_1))(Y^q_{i_3}(t_2) - Y^q_{i_3}(t))(Y^q_{i_4}(t_2) - Y^q_{i_4}(t))] \tag{14}$$

$$\le d^2\sum_{\mathbf{i}\in[[1,n]]^4}(i_1i_2i_3i_4)^{-1/\alpha}|\mathbb{E}(\tilde{\varepsilon}_{i_1}\tilde{\varepsilon}_{i_2}\tilde{\varepsilon}_{i_3}\tilde{\varepsilon}_{i_4})|D_{\mathbf{i}}(t,t_1,t_2) \tag{15}$$

where

$$D_{\mathbf{i}}(t,t_1,t_2) = \mathbb{E}|Y_{i_1}(t) - Y_{i_1}(t_1)\|Y_{i_2}(t) - Y_{i_2}(t_1)\|Y_{i_3}(t_2) - Y_{i_3}(t)\ \|\ Y_{i_4}(t_2) - Y_{i_4}(t)|.$$

Consider $\sim_{\mathbf{i}}$, the equivalence relation on $\{1,\dots,4\}$ defined by

$$j\sim_{\mathbf{i}} j' \quad \text{if and only if } i_j = i_{j'}.$$

Let $\mathcal{P}$ be the set of all partitions of $\{1,\dots,4\}$ and $\tau(\mathbf{i})$ be the partition of $\{1,2,3,4\}$ given by the equivalence classes of $\sim_{\mathbf{i}}$. We introduce these definitions because, since the Y_i's are i.i.d., the term $D_{\mathbf{i}}(t,t_1,t_2)$ depends on $\mathbf{i}$ only through the associated partition $\tau(\mathbf{i})$. For example, if $\tau(\mathbf{i}) = \{1,2,3,4\}$, i.e. if $i_1 = i_2 = i_3 = i_4$, then

$$D_{\mathbf{i}}(t,t_1,t_2) = \mathbb{E}|Y_1(t) - Y_1(t_1)|^2|Y_1(t_2) - Y_1(t)|^2.$$

Or if $\tau(\mathbf{i}) = \{1\}\cup\{2\}\cup\{3\}\cup\{4\}$, i.e. if the indices $i_1,\dots,i_4$ are pairwise distinct, then

$$D_{\mathbf{i}}(t,t_1,t_2) = (\mathbb{E}|Y_1(t) - Y_1(t_1)|\mathbb{E}|Y_1(t_2) - Y_1(t)|)^2.$$

For $\tau\in\mathcal{P}$, we denote by $D_\tau(t,t_1,t_2)$ the common value of the terms $D_{\mathbf{i}}(t,t_1,t_2)$ corresponding to indices $\mathbf{i}$ such that $\tau(\mathbf{i}) = \tau$. Define also

$$S_{n,\tau} = \sum_{\mathbf{i}\in\{1,\dots,n\}^4;\tau(\mathbf{i})=\tau}(i_1i_2i_3i_4)^{-1/\alpha}|\mathbb{E}(\tilde{\varepsilon}_{i_1}\tilde{\varepsilon}_{i_2}\tilde{\varepsilon}_{i_3}\tilde{\varepsilon}_{i_4})|.$$

With this notation, Eq. (15) can be rewritten as

$$\mathbb{E}|\widetilde{Z}_n(t_2) - \widetilde{Z}_n(t)|^2|\widetilde{Z}_n(t) - \widetilde{Z}_n(t_1)|^2 \le d^2\sum_{\tau\in\mathcal{P}}S_{n,\tau}D_\tau(t,t_1,t_2). \tag{16}$$

Under conditions (2) and (3), we will prove that for each $\tau\in\mathcal{P}$, there exist $\beta_\tau > 1/2$, a nondecreasing continuous function F_τ on $[0,1]$ and a constant $S_\tau > 0$ such that

$$D_\tau(t,t_1,t_2) \le |F_\tau(t_1) - F_\tau(t_2)|^{2\beta_\tau}, \quad 0\le t_1\le t\le t_2, \tag{17}$$

and

$$S_{n,\tau} \le S_\tau, \quad n\ge 1. \tag{18}$$

Eqs. (16)–(18) together imply inequality (12) for some suitable choices of $\beta > 1/2$ and F.

It remains to prove inequalities (17) and (18). If $\tau = \{1,2,3,4\}$,

$$D_\tau(t,t_1,t_2) \le \mathbb{E}|Y_1(t) - Y_1(t_1)|^2|Y_1(t_2) - Y_1(t)|^2 \le |F_2(t_2) - F_2(t_1)|^{2\beta_2}$$

and

$$S_n^\tau = \sum_{i=1}^n i^{-4/\alpha}\mathbb{E}\tilde{\varepsilon}_i^4 \le C(\alpha,4).$$

If $\tau = \{1\}\cup\{2\}\cup\{3\}\cup\{4\}$, the Cauchy–Schwartz inequality entails

$$D_\tau(t,t_1,t_2) \le (\mathbb{E}|Y_1(t) - Y_1(t_1)|\mathbb{E}|Y_1(t_2) - Y_1(t)|)^2 \le |F_1(t_2) - F_1(t_1)|^{2\beta_1}$$

and

$$S_n^\tau \le \sum_{\mathbf{i}\in\{1,\dots,n\}^4;\tau(\mathbf{i})=\tau}(i_1i_2i_3i_4)^{-1/\alpha}|\mathbb{E}\tilde{\varepsilon}_{i_1}\|\mathbb{E}\tilde{\varepsilon}_{i_2}\|\mathbb{E}\tilde{\varepsilon}_{i_3}\ \|\ \mathbb{E}\tilde{\varepsilon}_{i_4}| \le C(\alpha,1)^4.$$

Similarly, for $\tau = \{1, 2, 3\} \cup \{4\}$,

$$\begin{aligned} D_\tau(t, t_1, t_2) &= \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)| \mathbb{E}|Y_1(t_2) - Y_1(t)| \\ &\le |F_1(t) - F_1(t_1)|^{\beta_1/2} |F_2(t_2) - F_2(t_1)|^{\beta_2} |F_1(t_2) - F_1(t)|^{\beta_1/2} \\ &\le |(F_1 + F_2)(t_2) - (F_1 + F_2)(t_1)|^{\beta_1+\beta_2} \end{aligned}$$

and

$$S_n^\tau \le \sum_{1 \le i \ne j \le n} (i^3 j)^{-1/\alpha} \mathbb{E}|\tilde{\varepsilon}_i|^3 |\mathbb{E}\tilde{\varepsilon}_j| \le C(\alpha, 3) C(\alpha, 3).$$

or for $\tau = \{1, 2\} \cup \{3\} \cup \{4\}$,

$$\begin{aligned} D_\tau(t, t_1, t_2) &= \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 (\mathbb{E}|Y_1(t_2) - Y_1(t)|)^2 \\ &\le |F_1(t) - F_1(t_1)|^{\beta_1} |F_1(t_2) - F_1(t)|^{\beta_1} \\ &\le |F_1(t_2) - F_1(t_1)|^{2\beta_1} \end{aligned}$$

and

$$S_n^\tau \le \sum_{1 \le i \ne j \ne k \le n} (i^2 jk)^{-1/\alpha} \mathbb{E}|\tilde{\varepsilon}_i|^2 |\mathbb{E}\tilde{\varepsilon}_j \parallel \mathbb{E}\tilde{\varepsilon}_k| \le C(\alpha, 2) C(\alpha, 1)^2.$$

Similar computations can be checked in all remaining cases. The cardinality of $\mathcal{P}$ is equal to 13.

Step 5. We prove Theorem 1. For each fixed $t \in [0, 1]$, Kolmogorov's three-series theorem implies that $\widetilde{Z}_n(t)$ converges almost surely as $n \to \infty$. So the finite-dimensional distributions of $(\widetilde{Z}_n)_{n\ge1}$ converge. The tightness in $\mathbb{D}^d$ of the sequence has already been proved in step 4, so $(\widetilde{Z}_n)_{n\ge1}$ weakly converges in $\mathbb{D}^d$ as $n \to \infty$. We then apply Theorem 1 of Kallenberg (1974) and deduce that $\widetilde{Z}_n$ converges almost surely in $\mathbb{D}^d$. In view of step 1 and step 2, this yields the almost sure convergence of the series (1). □

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